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ABSTRACT

This report documents events and activities that took place in the course of meetings and seminars investigating the cognitive processes and learning styles of elementary school teachers. The report includes an overview and six separate documents, each presenting one facet of the project. The overview is followed in section two by a proposed methodology for the micro-analysis of spontaneous learning as it occurs during the process of building a musical tune with arrangements of bells. Section three deals with "critical moments"--events which are spontaneously recalled and used as images for understanding new situations. Section four is a longitudinal analysis of the evolution of an "idea" in the course of the work of a group of teachers in elementary arithmetic. Section five focuses on the role of interpersonal relations among teachers and between teachers and staff as these influenced the outcomes of seminar explorations and discussions. The sixth section describes the outcomes of the project as seen in observations of five of the teachers' classrooms, and section seven presents brief sketches of each of the participating teachers in the light of their participation in the project. (JD)

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# ANALYSIS OF DATA FROM AN EXPERIMENT IN TEACHER DEVELOPMENT

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with

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and
Magdalene Lampert

SP 028 035

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# FOREWORD

This report is a sequel to the "Final Report on An Experiment in Teacher Development" submitted to the National Institute of Education in March, 1981 under Grant Number G-78-0219. We assume that the reader is familiar with, and has at hand, that previous report since we refer to it from time to time in the present one. In particular, we direct the reader to page 12 of the earlier report where we discuss the notion of "giving a child reason." "Giving reason" played a central role in the teachers' learning and also, later, in their work with children in the classroom. It continues to be referred to in the current analysis.

This Report was prepared under NIE Grant Number G-81-0042 We wish to thank Rene Gonzales, our Project Officer, for his enthusiastic support for the Project and for his patience during the writing of this report.

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# PREFACE

This report includes six separate documents in addition to an overview, each of them presenting one face of our work: The overview is followed, in Section II, by a proposed methodology for the micro-analysis of spontaneous learning as it occurs during the work of making something (building a tune). Section III goes on to propose analysis on a larger scale by identifying "critical moments" throughout the seminar sessions. Section IV is a longitudinal analysis of the evolution of an "idea" in the course of the work of one group of teachers (elementary arithmetic). Section V focusses on the role of interpersonal relations among teachers and between teachers and staff as these influenced the outcomes of our work together. Section VI describes the outcomes of the project as seen in observations of five of the teachers' classrooms, and Section VII presents brief sketches of each of the teachers in the light of their participation in the project.

The Sections also reflect their authors' differences in style, approach and choice of materials. Sections II and III were prepared by Jeanne Bamberger who concerns herself more with methodology of analysis and perhaps a foray into what might pass as theory. Eleanor Duckworth, who prepared Section VI, makes a more intimate, extended, richly documented account and interpretation of actual events as they occurred over the second year of the seminar. Section V was prepared by JoAnne Gray (with editing by Bamberger and Duckworth) who was not herself, a participant in the project; she was thus able to take a more distanced view as she looked in at us (through the video-tapes) from the outside. Finally, Magdalene Lampert, in Section VI, reports from her view as observer in five of the teachers' classrooms, summarizing some of what she saw there, and what she heard in subsequent interviews with these teachers.

# I OVERVIEW

As the preface suggests, we have had to take into account in this analysis the vast and varied data accumulated during the course of the project itself: 180 hours of video-taped sessions with two groups of teachers; logs written by the teachers as personal reflections on the seminar sessions and on their lives both inside and outside the classroom; notes taken by staff members; reports from the adjunct teacher who worked in the classrooms of 6 of the participants; interviews with individual teachers and anecdotal reports of informal meetings between teachers and staff members. While we have worked to give a fair perspective on this mass of data in the report, we foresee that the work of disseminating the results may continue for some time to come and may also continue to take different forms. For example, one Ph.D. dissertation has already grown from the project, several published articles have appeared, there have been a number of addresses at various conferences* and we anticipate that a book will eventually grow out of our work.

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* See Appendix.

The task of analysis has been particularly demanding because from the outset we committed ourselves to staying close to the "phenomena" of the teachers' experiences. This has required us to develop modes of analysis that will document accurately. To do so we have made fine-grained descriptions through close analyses of actual protocols and through long and careful studies of video-taped sessions. Through these we believe we have succeeded in capturing the complex, multi-layered texture of the data that the project has generated. In doing so, we have also tried to include the emergence of new ideas, views, and feelings as these occurred in context.

The multi-layered texture of the data is evidenced in a number of ways: the seminar sessions included a wide diversity of experiences including making experiments with materials as varied as music, physics, the solar system and arithmetic. Further, through the discoveries the teachers made in doing these experiments, they also learned that they could learn "on their own steam." At the same time, they learned how to look at the kinds of learning with which they were involved. In this process the participants developed the ability to learn from one another especially as this involved getting inside one another's minds (see Sections 4 and 5 of this report). For example, they learned that new questions and new insights could emerge as a result of someone else's questions and insights. They became supportive of one another's confusions and fears, and in doing so, the participants were able to break through the isolation that they, like most teachers, experience: They discovered that they had shared puzzles, feelings of success and lack of it, and sometimes deep doubts about themselves as teachers and even about "school" as an enterprise. Finally the participants learned *with* one another to face risks--the risks, for instance, of open-ended activities such as we asked them to participate in and the risks of exposing their various beliefs and ideas to scrutiny--their own as well as each other's.

This diversity along with its complex, multi-layered texture also signals another important result of the project: To achieve the kinds of goals that we did and, in any case, towards which we were aiming, *takes a great deal of time*. That this is true is evidenced by the fact that it was primarily among the teachers who participated in the project over two years (between 45 and 60 sessions) that we saw really significant learning and growth. In contrast, among those in the second group who participated for only one year and then only every-other week (15 sessions), the extent of learning and growth was considerably less. And this even though the second group benefited from our previous experience during the first year.

This may seem like a troubling result in the view of possible applications of the project to, for example, in-service training programs in public school settings where even 15, 3-hour sessions might seem an extravagance. However, the need for extended time reflects, we believe, not only the depth of the re-thinking that the participants were willing to do during the seminar but more importantly, the complexity and profundity of the work of teaching and learning, itself, when [illegible] engage the sorts of thinking and reflection which the seminar encouraged.

This finding suggests that if we are to achieve the kinds of change in classrooms towards which we and others are striving, in-service programs should be available to teachers as a continuing process throughout their teaching careers. If school systems were to adopt such a proposal, then the pressures of time-constraints would be obviated since the processes that we developed, for example, would be on-going rather than a single, one-shot experience. Such in-service programs could become a revolving process whereby more experienced teachers could assume facilitator roles in relation to less experienced teachers with this responsibility passed around among the teaching staff. If the development of such programs were to be encouraged by NIE, it could be considered a really significant result of their investment in this project.

## Modes of Work in the Seminar

To characterize the process of learning as it occurred through the ways we worked together in the seminar, we have found it useful to borrow from D. A. Schön (1982) the notion of "learning as reflective conversation with materials." (See also Section II of this report.) "Conversation" is meant, here, both literally and figuratively. That is, it refers both to the literal, on-going conversations that took place between staff and teachers and among teachers, but it also refers to the "conversations" between teachers and their materials in the variety of specific experiments with which they became engaged. "Materials" as used in the context of the seminar sessions, varied from bells/pitches used for making a tune, balls and ramps for experimenting with speed and acceleration, to a story from one of the teachers' classrooms, a video-tape of a child engaged in some problem, to a teacher's question, confusion, or insight. Each of these materials was treated as something to reflectively act upon--to probe, perturb, to make something new of or with.

As a result, thinking and learning became more like a process of making something--a design for a building or a table, a pattern of colors/shapes, repairs to a bike, a car, a faucet. And as in making something, the materials are shaped and re-shaped as the maker "talks" to her materials through acting on them. In turn, as she attends to the "back-talk" of the materials which results from these actions, the maker finds new meanings in them.

Learning through such reflective conversation is in contrast with the means-ends instrumental logic that is usually associated with learning, thinking and problem-solving.

One of the positive outcomes of the project is the degree to which both kinds of conversations--literal and figurative--became actively reflective. For example, simply telling a story or expressing a view, on one hand, or manipulating materials in a ritual fashion by following rules, on the other, became, instead, occasions to question and probe. And to the extent that these conversations became reflective conversations, they also led to new insight or to the restructuring of a previously held view as an individual or the group came to see a story or some concrete materials in a new way (see especially, Section IV).

Learning took place, then, through the continuing accumulation of such experiences. For example, the accumulated interactions between the teachers and the staff--our on-the-spot actions, responses, questions and other interventions--served as a kind of living example rather than a specifically didactic "method." Indeed, the importance of the quality of staff interventions in creating an environment within which the work of the group took place, has gained in significance for us only during this last phase of our analysis. Earlier on it was, so-to-speak, transparent to our assessment of the data--i.e., we looked right through it as a factor in effecting positive change in the teachers' classrooms, in their views of what it might mean to "teach," and, indeed, of themselves as teachers. (In this regard, the "eyes" of other, more distanced observers has been most helpful (see Section V).

We found, for instance, that in our conversations during the seminar sessions, the staff stimulated active reflection by the kinds of "bootstrapping" we practiced--i.e., the informed but still usually unplanned/spontaneous questions and probes we made in response to a particular event or comment as it occurred. In turn, the participants gradually learned how to invent on-the-spot experiments to query their own responses to direct observations of some phenomenon in working on a problem, or to probe their observations of a child's behavior. We also encouraged more reflective conversation when the material was a child's question and what it might mean (Does Dataman have eyes?" "Is that the same Jesus as in church?"); when individuals found themselves confused in doing chip-trading ("I understand it, but I don't have the...I'd probably have to trade chips for another hour and write down everything before I could do that"); or when a directly perceived surprising result happened in making an experiment ("The distance between marks is awfully undifferent!"). This mode of reflective conversation was something we practiced together. It was, at best, what the teachers learned how to do and, most of all, learned that they could do. Suzanne, one of the teachers, reported that "having 'conversations' with kids is the most important overall idea from the project that permeates her teaching." (See Section VI.)

But it was never actually formulated. What we practiced existed as a form of action in the seminar and was also passed on to the teachers' classrooms in the form of their new actions. These varied with each individual teacher. What the teachers learned, then, they learned through their increasingly reflective interactions with the varied kinds and senses of materials that were present in the seminar sessions. The outcome, the "sediment" from these accumulating experiences was reflected in a gradually evolving change in the participants' ways of responding to and questioning a child's behavior, to making and using curriculum, to class management, to new ideas as these were encountered in the classroom (see Section

VI). But in the midst of doing it, it was often difficult for the participants to see what was happening or to say what was happening. The accumulating experiences became like beads on a common thread--images, stories, in-group phrases ("giving a child reason"). The accumulating experiences developed a kind of culture to which members of the group belonged.

## Research Revisited

Lest this sound more like a "cult" than a culture, let alone research, we hasten to say that our work has also forced us to consider deeply what "research" might mean given the purposes of the project and the nature of the data as evidence.

In Section II of the report we propose a new view of what might count as research in analyzing active learning, along with what might be considered evidence in such learning situations. We also propose, here, a new view of how the "informed observer" can go about coming to see phenomena in a participant's work that may at first be missed because it is orthogonal to the observer's initial assumptions concerning the task and even the field of knowledge of which the task is an instance.

Looking back at the original proposal, another rather surprising result emerged from this analysis of the data: The notion of "teacher-researcher" which had played a rather significant role in the original proposal, essentially disappeared once the real work of the project began. While we were certainly aware that we, as staff, had learned from the practices of Piaget, for example, in his clinical interview techniques, the uses to which we put such techniques in the seminar were not overtly associated with either Piaget or with research, as such. Instead, traditional notions of both research and teaching were reconstrued in reciprocal interaction with one another. And with these new constructions the rather stiff artifact, teacher-researcher, simply became irrelevent.

The new views as they developed in the seminar, stemmed from the emergence of a different relation between teachers and learners and between teachers and research. On this view, experimenting--probing, questioning, perturbing a learner's understanding/thinking--is a mutual enterprise in which teacher and learner are both active participants. Reflective conversation between teacher and learner and conversation between teachers/learners and materials are seen as ways of helping the learner to gain insight into his/her own understanding. And through this kind of mutual experimenting, the learner is helped to move beyond what and how he/she knows already, to achieve new understanding. At the same time, learners acquire tools for carrying out this kind of investigation ("research") on their own. For example, the reflective conversations between teacher and learner are coupled with their shared reflective conversations with materials, And through cumulative experiences of this kind, learners became accustomed to carrying on such conversations by themselves--talking back to materials by reshaping them while listening to the back-talk of the materials as a result of their actions on them. With this view, teaching, learning and research become a single, unified, enterprise.

## II A METHODOLOGY FOR RESEARCH INTO LEARNING

In this section we describe a methodology that proved useful in understanding the teachers' work in specific activities such as tune-building, and everyday-physics experiments. In developing the methodology we have built on the notion of learning as a process of "reflective conversation with materials" through which individuals "come to see in new ways."* In retrospect, we have found that this way of viewing learning describes much of how we worked with the teachers in the seminar sessions, the mode of learning they practiced in their work in building and experimenting, and also the kind of learning that they brought to their work with children in their classrooms as this was influenced by the project.

This section differs from the others in this report in that it is not primarily about the work of the teachers, themselves. Rather, it suggests a theoretical framework that we brought to our analysis of this data from the view of research into conceptual change. While the example we use to illustrate this theoretical approach is drawn from the work of two teachers in the project, it has, we believe, implications that extend well beyond the project, itself. As a possible contribution to a more general framework for protocol analysis and for educational research, it is, then, a significant outcome of the project.

In the course of analyzing the work of teachers in the project, we set ourselves the task of trying to capture moments in which individuals actually came to see in new ways. To do this, we chose some of the simple but rich task situations we had video-taped in the work of teachers working in small groups. Some of the tasks we borrowed from more traditional ones (like the Vygotsky block task) but some were more open-ended (like making a tune). Our real work began when we faced the problem of making sense of what our task-participants were doing.

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* See also Schön (1982) and Bamberger and Schön (in press). This section was written in collaboration with D. A. Schön and, in a revised version, will appear in the journal, Art Education.

The modes of analysis we suggest here evolved over a number of tasks in which the teachers were involved and also over several years (1978-82), eventually raising questions that went far beyond the immediate objective of capturing moments of insight, i.e., of significant learning. The most powerful strategy we found as a starting point for our analysis was something we called "chunking the protocol." This involved looking for what seemed important boundaries that articulated observable phases or organic "chunks" within the continuing course of a participant's work. These we thought might signal shifts in behavior and/or focus in the evolution of the participant's understanding.

On the first several passes over the tape, we searched for such boundaries without trying to be explicit about the criteria we were using or exactly what sorts of behavior were signalling the boundaries we found. We simply tried to mark "something new happening." Once having found a chunking that seemed right, we went back and looked for the criteria we had quite spontaneously used. In other words we reflected on our own behavior as observers while at the same time letting the behavior of our participants "talk back" to us in the context of what we saw as possibly significant turning points in their work. We asked, given this chunking, what are we taking to be "something new" and why? What does this tell us about what we take to be the purpose of the task, or, indeed, what we take even to be "seeing" in some way, let alone a new way?

This initial chunking led to others where we now explicitly set criteria different from the ones we had found ourselves using in making the first chunking. We also showed the tape to other persons whom we expected to bring a different "set of eyes" to the protocol, working with them to help us determine the criteria they were spontaneously using.

These various chunkings served to help *us* see in new ways. New moves, new behavior, new features of the protocol were "liberated"--i.e., things we hadn't seen at all became "visible." But most important, we gained insight into our own, often tacit assumptions concerning the nature of the task, the structure and theory of the task-domain, and even the possible elements and relations that we had taken to be "givens" in the materials.

The result was a collection of descriptions that might resemble a series of maps, all of the same terrain, but with the delineation of points and parts and the lines and shadings that did so, all quite different depending on what the cartographer was paying attention to as significant things (topography, roads, weather). And just as each of these maps is "right" in its own terms, so we took each of our chunkings of the protocol to be "right" in *its* own terms. The problem in our case was to be convinced that we had correctly discovered assumptions-in-use underlying our various chunkings of the protocol. For these, in turn, would at least guide what we considered a new way of seeing on the part of a participant.

In addition, we needed to find ways of coordinating these multiple views so as to inform our ultimate search. But with multiple views available, we had a better chance of answering our initial question: what was the nature of the processes that led our participants to *learn*, in particular, to learn *through their own experience*, new knowledge that was not present at the outset of their work?

Plato puts this problem of learning "on your own steam" and recognizing "it" when you find it, into the mouth of Meno (in the dialogue of that name):

> But how will you look for something when you don't in the least know what it is? How on earth are you going to set up the object of your search? To put it another way, even

> if you come right up against it, how will you know that what you have found is the thing you didn't know? (Meno 80.D)

And after much discussion, Socrates says (in a tribute to "discovery learning"):

> ...one thing I am ready to fight for as long as I can in word and act: that is, that we shall be better, braver and more active men if we believe it right to look for what we don't know than if we believe there is no point in looking because what we don't know we can never discover. (Meno 86.C)

This process we were able to see among our participants: Looking for something they did not yet know, they most often found it. And most often it emerged through a "piton effect," like the process of pulling yourself up through your own power to a new position (and view) on a mountain. The question then becomes, what are the pitons that our participants used in the course of their work to achieve a new view of their materials and tasks? It was, to pursue our metaphor, because of the nature of the pitons we found that we turned from thinking of "making" as a cognitive process to a view of cognitive processes as a kind of "making." We found our participants improvising, we found them engaged in on-the-spot experimenting in response to the new phenomena they were discovering. In short, we found them "conversing" with their materials. Their conversations were more like the making and shaping of coherence in the arts than like the means-ends, instrumental logic usually associated with puzzles and problems. The example to which we will turn in a moment illustrates the rather unexpected nature of the events that seemed to lead our participants beyond what they knew and even to recognize what they found as "the thing they didn't know."

In order to capture the sense of movement and instability associated with, even necessary to learning and change, we have coined the term, knowledge-in-action (KIA). By KIA we mean the current state of an individual's possible mental constructions for shaping coherence with respect to some present phenomena. We use the term in place of more traditional expressions such as "internal representation," or "cognitive schemas" in order to capture this sense of continuing mobility. We also want to suggest by the term that KIA need not be associated with a capacity for external symbolic expression. Thus KIA may often be that which an individual knows how to do but cannot yet say.

However, we also intend to include in KIA, individuals' current capacities for making descriptions in the given domain as well as in others. This because there are often important interactions between individuals' KIA with respect to making descriptions and their current capacities to act on, recognize and construct coherence within the materials of a present situation. This interaction between materials and modes of description will in fact play a major role in what we have called the "piton effect."

## Conversational Learning

The task we have chosen involves two of the teachers in the work of making a tune. The two participants (who worked together on the task) were given five Montessori bells* as their building materials. The particular bells/pitches were selected by us before-hand so as to make the task intriguing and also somewhat problematic (more on this, below). The instructions, as they were actually given to the pair, (and to other pairs of teachers in the group) were as follows: 1) make a tune that you like using all the bells in the collection you have been given; 2) make as rich a description as possible of your completed tune; 3) write a set of instructions so someone else could play your tune on your bells.** Building the tune was an open-ended task in that there was no "right answer" and particularly because the criteria for success depended on the participants, themselves. We should say right off that these teachers (as well as all the others in the group) were all able to build tunes that, in fact, made sense to them as well as to others. Thus, their criteria for coherence went beyond mere personal "taste." Our interest, then, is not in whether or not the participants were able to complete the building task, or whether they could do it at all (they all could), but rather, the evolutionary course of their work--the moves they made, the strategies they used, and most of all, the ways in which general, shared criteria for coherence emerged.

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* Montessori designed a set of bells to be used in her classroom as one of the "sensorial materials." Each individual mushroom-shaped metal bell stands on a wooden stem, attached to a small wooden base. The bells are tuned so as to play different pitches. However, unlike most other pitch-playing materials, all the bells look the same so that differences in pitch are distinguishable only by actually playing on the bells. Montessori, in her wisdom, was thus able to focus children's attention on "pitch-sense," alone, without cues from size, shape, or position in an array.

** Only the first of these will be discussed in this analysis.

We have chosen this protocol as an example because it seems a paradigmatic instance of what we have come to call "conversational learning." By this we mean the gradual evolution of making something (a tune, an insight into a child's questions, a new understanding of place value) through reflective "conversation" between makers and their materials in the course of shaping meaning and coherence. "Reflective" has at least two intentions here and often they are so intertwined as to be indistinguishable: the makers' spontaneous (and active) reflective response to their actions on the materials, and the "reflection" of the materials (as they take various shapes and forms) back to the makers. The two kinds of reflection can be thought of as two kinds of "talking back." In the first the makers talk back to the materials (re-shaping them), in the second the materials talk back to the makers, re-shaping what the makers know about them. The distinction is, in a sense, moot since materials don't "talk" and the "talk" of the makers is most often (but not always) action rather than words. Further, the back-talk of the materials is only to the extent that the makers "hear" it--i.e., the current state of the makers' knowledge-in-action strongly conditions what they recognize and apprehend as the "message" implicit in the current state of the materials. In turn, the makers' talk back to the materials is not "heard" by the materials except to the extent that these are re-shaped in some way.

The matephor of "conversation" can serve a two-fold purpose: it can, as suggested above, serve as a way of viewing the evolving course of our participants' work; at the same time, it can serve as a way of setting a mood, an ambiance for research. In this latter sense, the notion of "conversation" becomes important and productive as it encourages us, as researchers, to make our own action experiments in search of the emergence of new meaning, new features, new structures. For example, as in the process of multiple chunkings of a protocol discussed earlier this means being vulnerable to the possibilities of restructuring our assumptions: what we take to be the "givens" of the materials and indeed, the "givens" implicit in the theory of the domain. At the same time, such an approach requires that we meet the challenge of rigorously testing the validity of emerging new structures against the observable evidence of the protocol.

The question of evidence becomes central since what constitutes evidence is necessarily influenced by the knowledge-in-action that we, ourselves, bring to the situation. We are, then, sensitive to the problems of the "hermeneutic circle"--that is, our interpretations of the protocol as "text" are dependent on what we are able to see, the filters built in to our apprehension; the participants' interpretation of the task, its materials and its criteria for success are also dependent on what they are able to see, the filters built into their apprehension. At worst this sensitivity results in despair, something like the despair expressed by Meno. We aim then at something more like a "hermeneutic spiral" which moves dialectically through possible interpretations moving out and beyond, rather than a closed circle where head and tail forever meet going nowhere.

## Making Meaning

If we take seriously the notion that meaning, itself, is a process of making, describing the work of the participants presents certain difficulties at the outset. For example, if we name the pitches of the bells the participants were given with their conventional letter names (D, G, etc.), the reader is already privy to information that the tune builders were not. More importantly, such information is quite different in kind from that of the players, including a whole set of underlying assumptions that we cannot appropriately attribute to the participants as they begin their work. More importantly, giving the reader the names of the bells/pitches along with the meanings they carry, puts us in danger of reading back onto the participants' moves--their decisions and actions--"givens" that are ours but not theirs. At the same time we run the risk of failing to recognize and to appreciate what it is the participants can do and the cognitive work involved in their on-the-spot constructions and decisions.

The problem points up the powerful role of learned and internalized structures associated with symbols and categories of analysis as these influence the way the "informed observer" actually comes to see, and in this case, to hear. In short, it points up the kind of problem we faced in our continuing "conversations" with the data of the protocol.

Recall that unlike musically trained tune-builders, our untrained participants have no way of placing and thus naming the bells/pitches with respect to a constant, fixed reference such as a scale, a key, or units of measure for identifying the intervals among and between them. As a result, the participants must give meaning to the elements by making a coherent universe within which they gain meaning--a universe that can include them. For example, while the goal of the participants is to make a tune, the evolution towards this goal includes making a number of "transitional objects"--namely a series of constructed and reconstructed bell-arrangements on the table. These transitional objects in their various transformations serve as the intermediary between the process of shaping coherence, itself, and the final tune. At the same time the arrangements of bells on the table also serve to "hold still" the meanings the participants give to the bells. Each arrangement becomes a reference for these meanings. And since each arrangement is also a concrete entity, we call each transitional object a reference entity--an embodied and enacted description of what the participants know so far. The meaning and functions of the bells/pitches evolve in reciprocal interaction with the construction of the reference entities. As such they serve as a materialized "log" of the making process--a series of sketches.

The term reference entity is meant to contrast with reference structure which we take, for example, to be available to musically trained individuals. By reference structure we mean that complex network of internalized mental structures which is closely linked to the conventional symbol systems associated with a domain--here, music. We propose that conventional symbols serve as the intermediary between such internalized mental reference structures and the stuff of the domain. Indeed, we would argue that the notations determine, in important ways, just what are taken to be possible entities and relations in the stuff. In turn, the conventions of notation play a significant role in the development and the nature of such internalized reference structures. Thus, there is a reciprocal interaction between the ontology implicit in the notation system, the ontology inherent in the reference structures, on the one hand, and, on the other, what is taken to exist as phenomena in the domain. In the context of a reference structure, then, names name properties or relations represented in internal reference structures and found, by instantiation, in the phenomena of concern.

By contrast, the type and function of naming is noticeably different in the case of a reference entity. A reference entity serves a naming function but it does not literally name. That is, a reference entity is unique, often transient and it is "held" by the materials used for making things within the domain. A reference entity serves to single out, externalize and hold for current attention some emergent object or relation. For example, the position of a bell in the array can, for the builder, serve as a way of referring--most often as a way of referring to a kind of thing that is not defined with respect to its explicit inner properties or relations. The position of a bell, or indeed, a whole ordering of bells, functions, at some moment, to "dub"*--to stand for, call up, or point to--some relation or property not yet differentiated or defined but recognized. Further, the same positioning can refer at this moment to one property, at that moment, to another. Thus, a reference entity serves the function of on-the-spot naming within, and as part of, the making process.

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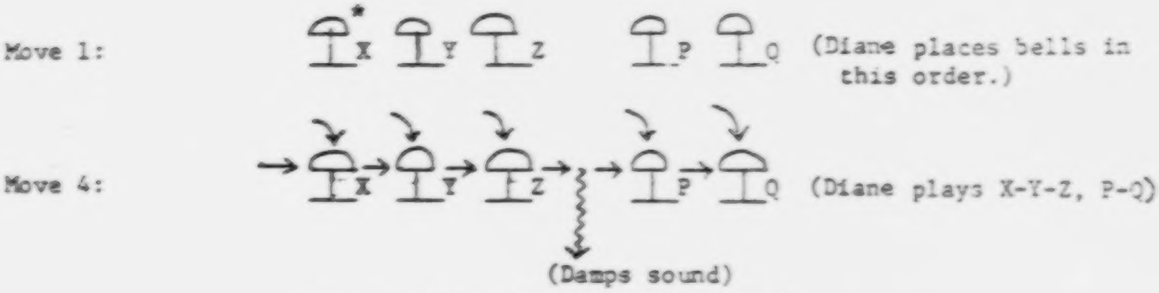
* For more on "dubbing" see Boyd, R., "Metaphor and Theory Change" (1979).

## Snapshots from the Protocol

We include, here, just three snapshots, or better, three "stills" from the participants' work. All of them are from relatively early in its course: moves 1 and 4, moves 8-10 occurred in the first 5 minutes; moves 27-30 occurred about five minutes later in this 45 minute protocol. Each of these glimpses share important characteristics: the reference entity is transformed in its use, in what it describes, and/or in its shape. In addition each example includes several media of description--space, gesture, sound/time, words.

In these first moments of the participants' work we see them engaged in a series of continuing experiments. Their questions seem to be: What have we got here; what sense can we make with it? We will argue that as the participants shift in their uses of the materials and with these shifts also in their modes of description, new features and relations are "liberated."

SNAPSHOT 1:



Diane: "They don't go together."

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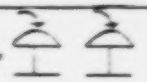
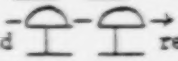
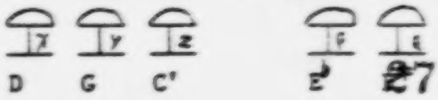
* stands for bells on the table,  represents playing on them and  represents the order in which they were played. We use simply letters as names for the bells rather than naming them with the pitches they, in fact, played, in order to help the reader stay closer to the tune-builders' experience. For those who want to speculate on the musical implications, the pitches are:



The tape begins after Diane has been at work by herself for a few moments casually experimenting with the bells. At Move 1, Diane, using the materials as objects to be moved about in space, makes a pattern consisting of two groups of bells/pitches The inner ordering of the bells and the space between them enactively describe the results of her initial experimenting. Diane has made a first reference entity, holding still what she has found so far: there are two kinds of bells--X Y Z is one kind, P Q another.

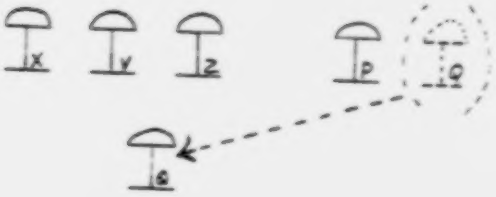
At Move 4 Diane makes another experiment and in doing so changes the use of her still-ordered materials: the bells, initially arranged as reference entity, can also be something to play on--a unique instrument that makes a single "tune." Using the reference entity now as instrument, Diane plays through them, left to right, "straight ahead." Her action path mirrors in gesture the structure of the static spatial reference entity: a spatial gap marking the two groups becomes a gap in sound and time (she damps the sound of X-Y-Z, making a silence and a pause). Finally, in response to the back-talk of her actions on the materials, Diane tells Carol (and us) in words what she apprehends: the two groups of bells "don't go together."

In these two moves, Diane develops a beginning repertoire for using her materials to experiment. The repertoire derives from the commonplaces of what she knows how to do already--her KIA: objects can be moved about and organized in space; these objects, when ordered, can also make an instrument to play on. Moving from one use to the other, the back-talk of the materials can, to begin with, test and confirm.

But spontaneous shifts from one use of the materials to another can do more: moves 8-10 produce something new.

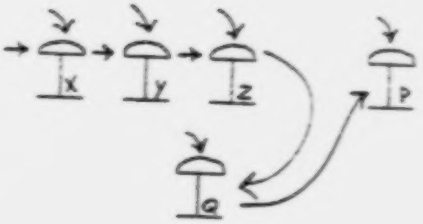
SNAPSHOT 2:

Move 8:



(Diane moves Q-bell forward.)

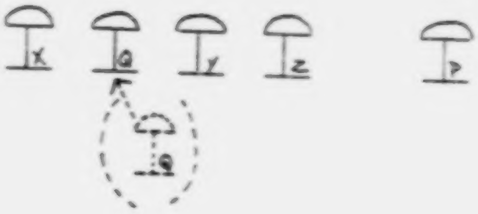
Move 9:



(Diane plays: X-Y-Z-Q-P)

Diane: "A doorbell."
Carol: "Or they belong to a different set."

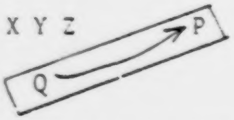
Move 10:



(Diane moves Q-bell between X and Y.)

Diane: "As far as we're going to put them in order."

At 8, Diane moves Q forward--the reference entity becomes a "workplace" with Q as the object of attention. And this new configuration, made to describe, to hold for attention, creates a new instrument, too. At 9, tracing the reference entity as a path, Diane plays on her new instrument. As she does so, the spatial boundary between Z and Q becomes a perceptual boundary; Q-P pops out as a contained, sounding entity.



The shift from materials as reference entity to materials as instrument spawns the emergence of something new, the figure Q-P--a found object within the pitch collection. Diane recognizes it and names it: "doorbell." Using the bells first as reference entity to describe and then as instrument to play on, Diane unintentionally discovers within the materials a surprising new object. This unplanned move serves as the piton whereby her reflective conversation carries her beyond what she knows already.

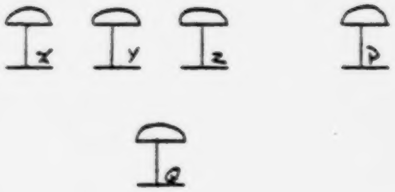
But Diane has also popped out of the universe she is trying to construct: "doorbell" gains its coherence as it represents (calls up) a useful object in the everyday world. Multiple descriptions are not always helpful. Carol brings the tune-builders back to the task at hand. For Carol, P and Q "...belong to a different set." A figure (Q-P) is given meaning only in terms of what you see it as.

At 10, perhaps triggered by Carol's comment, Diane makes a new configuration of bells, gives a new status to Q, and also changes the reference of the reference entity. She moves Q between X and Y (making X-Q-Y-Z), on the criterion "...put them in order." "Order" means (she explains later) from low to high in pitch. Invoking this criterion, Diane gives Q a new meaning: no longer associated with P, Q is now seen as higher than X, lower than Y. And the group X-Q-Y-Z gains particular coherence as a progression of pitch properties ordered along the dimension low-high.

Between Move 10 and 27 Diane reviews for Carol her earlier moves. In the process the bells are re-arranged several times. Going on, Moves 27-30 were as follows:

SNAPSHOT 3:

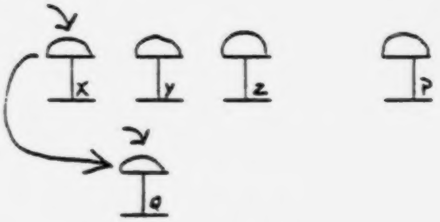
Move 27:



(Arrangement at Move 27.)

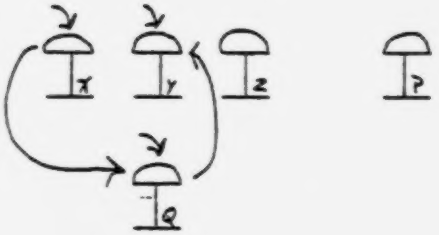
(Diane: "Try these two together (points to X-Q.)

Move 28:



(Carol plays X-Q.)

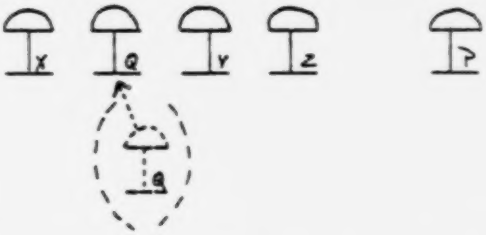
Move 29:



(Carol plays X-Q-Y.)

Diane: "Oh! That sounds nice!"

Move 30:



(Diane moves Q-bell between X and Y.)

At Move 27, the arrangement looks the same as at 10. Indeed, looking at Moves 27-30 the arrangement of bells seems to undergo the same changes as in Moves 8-10. However, both Q and the reference entity come to have new meaning--to refer in a new way. At 27 Diane again makes a workplace for experimenting--what can be done with Q? But at 28-29, following Diane's suggestion, Carol, using the reference entity now as instrument, traces a different path through it: X-Q, and then X-Q-Y. As Carol does so, Diane finds in the back-talk of the bells another surprising new object--a little tune figure that she likes: "Oh! That sounds nice!" Then, moving the bells once more to fix and describe, Diane again puts Q between X and Y. Now, using the materials as reference entity, the bells hold still in space the new-found tune-figure that "sounds nice."

But notice that in recognizing a "nice tune" Diane also *recognizes* a fresh criterion for coherence--one that was already "there" in her KIA as tacit norm, but it is only "liberated" in reflective response to a spontaneous instance of it. It is by invoking this new-found criterion that X-Q-Y gains new meaning as *figure* and Q is given new status as a legitimate member of it. Interestingly, although the reference entity now *looks* the same as at Move 10, it *refers* differently: at 10, X-Q-Y-Z described a set of pitch properties ordered from low to high; at 30, X-Q-Y described a "nice tune." Meanings change as the participants' repertoire for possible uses of materials unfolds--e.g., as reference entity transforms spontaneously into instrument; as bells transform unintentionally from pitch-making objects that can be ordered low to high to pitch-making objects that can make a "nice tune."

These "stills" from the tunebuilders' work seem striking examples of reflective conversation with materials. The various bell arrangements seen as transitional objects in the making process, and the evolving meanings and functions given to the Q-bell within this series of sketches, tells the story. Tracing the course of the errant Q-bell we see the various "dubbings" the participants have given to Q:

1. As paired with P to make two "kinds:"

X Y Z P Q (Moves 1 and 4)

2. As the focus of attention in a work-place:

X Y Z P (Moves 8-9)
Q

3. As a member of an ordered set of pitch-properties:

X Q Y Z P (Move 10)

---

* This little figure in fact becomes the germinal motive from which the final tune evolves.

4. As a functional member of a figure that "sounds nice."

X Q Y Z $\dot{P}$ (Move 30)

As the tunebuilders slip from one familiar use of the materials to another (moving them about in space, playing on them), one possible meaning, now another, is liberated (e.g., "in order" as low to high, or as "nice tune"). Each new meaning is, in turn, "held" by the materials as reference entity. But such shifts in meaning happen only when previously tacit norms are liberated--for example, the capacity to recognize what "sounds nice." The piton, then, that brings Diane and Carol to a new view is the move that, albeit unplanned, becomes a pivot, shifting their actions on, and uses for the bells, triggering the unexpected emergence of possible criteria for coherence. Improvising, uses elide into one another, new relations, new meanings emerge, and these, in turn, re-shape the makers' KIA with respect to the task. In this way the tune-builder's commonplaces of possible things to do with the materials become, at the same time, things to think with.*

## Conclusions

We have tried to illustrate through these brief glimpses into the tunebuilders' work, the process of how, on their own steam, they come to see in new ways. Through reflective responses to their own action-experiments, Diane and Carol find and recognize elements, relations and objects that, in some way, they knew already: "...for seeking and learning are in fact nothing but recollection," (Meno, 81.d). At the same time they come to see these materials in new ways--they are building a unique coherence. Unexpected insight evolves in the work of making, but makers tend only to see it when, through the evolutionary process of the making, itself, they can recognize it. And when they do, the transitional objects, the moves on the way seem to disappear. Practicing a kind of "historical revisionism," they attribute insight to the moment when it occurs, even finding in the moment a sense of certainty--of course, "we knew it all the time!"

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* The notion of "things to think with" is borrowed from Seymour Papert (Papert, 1981).

This is one important reason why we, as so-called informed observers, failed to see at first the pitons by which individuals pull themselves up to a new view. We too had to experiment, restructuring our categories of analysis and with them, the criteria for evidence. And in doing so we gradually realized that the same sort of pitons were operating not only in open-ended composition-like tasks such as Diane and Carol's, but also in more constrained tasks--Vygotsky blocks, physics problems, even computer programming. We learned how to pay attention, for example, to spontaneous shifts in the uses of materials, to the influence such shifts had on what were possible criteria for coherence--not as parentheses or by-ways, but as the source of liberating new features. Once having done so, we saw our participants making use of these same kinds of pitons in our apparently constrained and "logical" tasks, too. At first we saw only the acts that matched our assumptions, the rest fell in between. A finished product--a computer program that works, a proof that matches the canonical one--tends to "wipe out" in its clarity and logic especially when expressed in conventional symbolic notations, the conversations with the materials through which they evolved.

But these are only sketches, much work remains to be done. We have tried to make an ambiance for research, give some examples, and to pose some speculative questions. Nelson Goodman is willing to answer one of them: "Even if the ultimate product of science, unlike that of art, is a literal, verbal or mathematical, denotational theory, science and art proceed in much the same way with their searching and building." (Goodman, 1978, p. 107) And Ben Shahn puts that "same way" like this: "So one must say that painting is both creative and responsive. It is an intimately communicative affair between the painter and his painting, a conversation back and forth, the painting telling the painter even as it receives its shape and form." (Shahn, 1957, p. 49)

## III CRITICAL MOMENTS

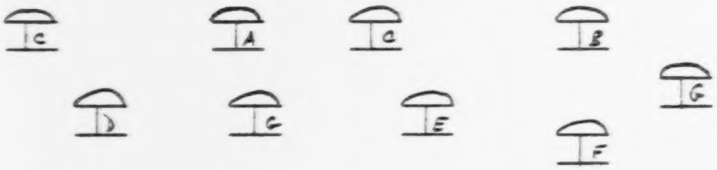
In addition to the methodology described in the previous section, we have begun to develop another approach to the analysis of our data. While this approach has not been fully developed as of this writing, we see it as a potentially promising one for the study of more global effects of the project. The idea here is to identify what we have called "critical moments" in the life of the seminar.

By "critical moment" we mean an event in the course of the seminar that is spontaneously recalled by one of the participants in the midst of a subsequent situation and used by her as an image for understanding that new situation. Thus a critical moment can only be identified, in fact only comes into existence, at the later moment when it is recalled and named--i.e., when a new experience is seen as a previous one ("It's the same thing as the 'building blocks.' It is, in fact. Not only is it like the 'building blocks' example, I mean, it is!").

Once a critical moment is identified (e.g., 'building blocks'), we can go on to ask: What is the named moment a name for? What is the 'family resemblance' between the first event and those in which it is recalled? How is the meaning given to the initial event changed as it is embedded in and influenced by the later events? By identifying and tracing the course of such critical moments, we can on a broad scale trace group-generated powerful metaphors over time, look at the learning that this helped to shape as well as the learning process itself. Taking critical moments as group-shared symbols, they become for the group, "things to think with." As such, they could constitute evidence for the intellectual content inherent in the cumulating experiences referred to above.

We have developed this form of analysis through two examples, one of which we will summarize here. The initial event occurred in Session 2 (October 4, 1978) and was recalled in Session 6 (November 28, 1978) and again several times in subsequent sessions. The "material" in Session 2 was a videotape of a child trying to build the tune, "Twinkle, Twinkle, Little Star" using Montessori bells. In order to provide the context within which the critical moment occurred, we must describe what the group had seen on the tape:

The child (Ricky) had been given 9 bells casually arranged on the table:

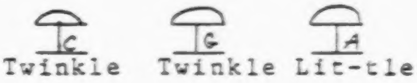


Each bell played a different pitch but there were two doubles--i.e., two G-bells and two C-bells. The bells were not labelled in any way and they all looked the same. (Labels are added in the above figure for purposes of explanation, here, only.) Ricky's procedure for building the tune had been as follows:

- Find a bell that could serve as the starting.
- Search among those left in the mixed array for the next bell/pitch in the tune.

- When the next bell is found, place it next to those already in the tune so as to build an accumulating row of bells on the table that will, when completed, play the whole tune.

The teachers were particularly interested by Rick's search strategy. They had observed that each time Rick went in search of a next bell, he played the tune through from the beginning. For example, when he had built the tune as far as

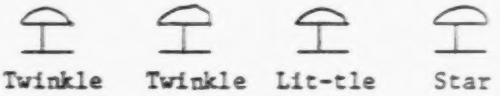


and was looking for the bell/pitch for "star", he would start from the beginning, play the tune as far as "lit-tle," and then test a new bell as a possible bell/pitch for "star." If the tested bell was not the one he was looking for, he would then repeat this whole process, starting over again from the beginning of the tune, only substituting a new test bell. This procedure continued until he found the bell/pitch he was searching for--in this case, the bell for "star."

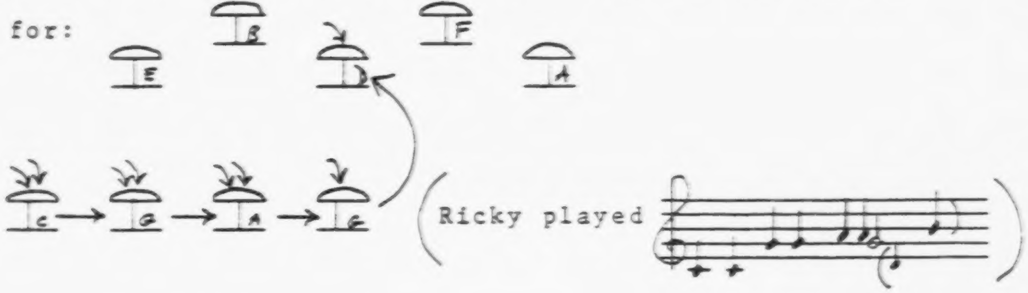
Preceding the "critical moment," the teachers were puzzling over the question, why did Ricky need to play the tune over again from the beginning of what he had built so far, each time he tried to find the next bell/pitch? The group had developed various possible "theories" in search of a plausible reason. In general, they saw Ricky's procedure as expressing an inadequacy on his part.

They had proposed such possible explanations as "lack of musical skills," "a weakness," "poor auditory memory," "a need for security."

After a good deal of discussion on this topic, Bamberger made an experiment with the group in order to see if they might change their views. She played a short portion of the tape again, starting at the point where Ricky had built the tune as far as:



The tape continued as Ricky went in search of the bell/pitch for "how." As usual, he started from the beginning of the tune but this time tested two bells (D-B) one after the other without starting over. Neither of his test bells was the one he was looking for:



At this moment, Bamberger stopped the tape. She then asked the group to sing the note that would correctly continue the tune--i.e., the note for "how." To their surprise, noone was able to do it!

After a long, rather uncomfortable silence, Bamberger urged the conversation forward by asking, in the light of the group's surprise:

Bamberger: Does that tell you anything about tunes?

LONG PAUSE

Bamberger: Does it tell you anything about why he needed to start over again?

After another long silence, Lee turned the conversation in a new direction:

Lee (rather quizzically): Actually, the fact that he kept going back, we all picked up and talked about; we stressed it a lot. But when Jessica and I did it (built the same tune) last week, we certainly went back and played the whole thing from the beginning.

Jessica: We hummed it.

Lee: We also hummed it. I was humming it in my head and I think I used that as the way I found the next note.

PAUSE

Duckworth: Could I ask you, Helen, about something you said earlier? When you were talking about his starting over again...because he hadn't 'mastered' it enough?

Helen: But it's all sequential, the repetition. That's why I thought it wasn't mastered.

Bamberger: So is it simply being repetitive or is it necessary?

Lee: Well, what Jessica and I said was that we were doing it also but in a slightly different way.

Bamberger: So he knows something about music without knowing that he knows anything about music. And so did you and so did Jessica.

Jessica: I would assume that no matter how much musical knowledge you had, you would probably still have to do that.

Carol: If you knew the third note was A, though...

Jessica: No, no, no, we're not talking notes. We're not talking names of things.

Lee: ...it's the relationship between the tones that counts, not the actual tone or its name...but it's the relationship between it and the one next to it and the one before it.

Jessica: You can't pull it out of space. You have to pretend...

Bamberger: So, can you think of any other situations where you get mileage out of starting over, or where you have to, in fact?

At this point, there was again a moment of silence, and then Lee made the following rather dramatic comment:

Lee: So in other words, all the discussion about weakness and learning mode and everything is basically down the drain because what we've just said is that nobody can do it any other way. Right?

Lee's comment was greeted with much rather nervous laughter--a kind of release of pent-up tension. The group's confusion together with the emerging new view was "unnerving." The intervention experiment and the discussion that followed had resulted in the participants realizing that they, too, in building the tune the previous week, had had to "play the whole thing over again each time"--either actually playing on the bells from the beginning or humming it "in their heads." The importance of playing from the beginning in order to establish a context, an orientation for search, was gradually emerging as a new, more positive way of looking at Ricky's strategy.

After a little more discussion, settling into this view, Jessica put another question, referring back to Lee's comment:

Jessica: But do you think...how many of us reached that conclusion? One of us verbalized it, but how many of us actually got there?

With this question the critical moment occurred. Helen, who had been sitting quietly during much of the previous discussion, responded to Jessica's question:

Helen: I realized when you were talking that music is building. You can't have the fourth block without the first. That came across just before Lee spoke [...down the drain] and then I realized that repetition in music must be necessary because

> you can't build--well, it's like a tower. And so I visualized it with kids in the kindergarten with blocks. But it came out only because of probing. I think you were...like pulling teeth.

With this, the group's mutual process, and, in particular, Helen's own thinking became (with Duckworth's help) the "materials" for reflective conversation--their own thinking "talked back" to them. Lee said:

| | |
|---|---|
| Lee: | I think we were trying to intellectualize a lot, and then the practical part came up when you [Bamberger] asked, "Well, what's the next note?" And everybody goes ahhh...not knowing what... |
| Helen: | My mold didn't fit when she [Bamberger] did that. What I was thinking all along didn't fit when you said that. So at that point I had to stay with my old mold and let you go, or I had to put my old mold back there and say, "Now, where can I go from here?" |
| Duckworth: | What was it about your old mold that didn't fit? |
| Helen: | Well, that I was set, that he couldn't have mastery of it because if he did he wouldn't have needed that constant. But then when we stopped it there and I couldn't [find the next note either], then I started to think about building. I still hadn't gotten to the point of realizing that you had to build. I was still somewhere floating, and then through the interactions [among the group], then it came...But it wasn't comfortable. I mean it wasn't comfortable...It was comfortable as soon as the interaction with you [Bamberger] and Lee went on and I was thinking of the blocks and I was hearing Lee verbalizing it and I was seeing the kindergarten blocks. And I've seen a kid build a tower, and I was saying, well, building, you can't...Then it was starting to get comfortable 'cause then I had a new...I could take my old mold and put it with the new... |

Helen's description is quite remarkable. While this was obviously an important insight for Helen, it occurred nearly at the end of the session and there was little further discussion about it. It was only later, when it was recalled in Session 6, that we discovered that Helen's insight had been a significant moment for others in the group, as well. But when the reference was made to "Helen's building blocks," it wasn't at all clear how the earlier incident made sense in the context of this later situation--i.e., it wasn't clear at first what "building blocks" meant in view of the concerns in the new situation: only with the help of the facilitator's probing did the meaning emerge.

The "material" in Session 6 was a computer music system that the group was exploring. They had been asked to think of the computer as a "kind of mind," albeit one made (programmed) by a person. By perturbing it, "asking" it to do various things, like play simple rhythms, the group could find out just "how it thinks." Well into this session, Jessica made a discovery concerning the time from one synthesized drum sound to the next that the computer "played":

| | |
|---|---|
| Jessica: | You can't tell how long it (one drum sound) is until the next one comes. (Pause) It's important! |
| Lampert: | Why? |
| Jessica: | I like those, you know...when they just kind of click. That's one of those music things. |
| Duckworth: | I don't understand. The enigma of it? |
| Jessica: | I like it because it puts into words what that experience is. It's the same thing as the building blocks. It is, in fact. Not only is it like the building blocks example, I mean it is. |
| Bamberger: | Say some more. Sounds interesting but...how is it? |
| Jessica: | I'm afraid I'm getting off the track. |
| Lampert: | No, I think you're on the same point. Then how is it like the building blocks example? |
| Jessica: | Ya know, way, way back...I responded positively to that statement because I like those things that click like that about music. And the building blocks was (uhhh)--and that's another thing--that |

you don't know the time of this note until you hear the next note. Then you go, "oh, my god!"

Duckworth: Building blocks...?

Jessica: Yeah.

Duckworth: The tune needs the beginning to rest upon? Helen's building block analogy? So this seems like a kind of neat musical insight you just got?

Jessica: It's an easily verbalized something that must be part of everything that's played, right? Then you have this little five word thing that tells you something that's always operating in music.

Helen (who had been very quiet until now): It's something you know but you didn't know 'til somebody verbalized it. Then you say, "Aha! That's true!"

Jessica (agreeing): It's one of those things that makes a part of experience that clear...in 5 words. It doesn't happen all that often.

From this episode, at least one thing seems clear: it was not the physical properties of "building blocks," as such, that were the relevant dimension of meaning that Jessica attached to that name. The name was an emblem, a talisman for something else. Jessica had apparently carried over from the original event "way, way back..." some meaning that she named "Helen's building block example," and this meaning surfaced in association with events in the new situation. But just what carried over? In what respects did Jessica see her insight in the new situation as like Helen's "building blocks example?"

To understand what Jessica could mean--what meaning she subsequently gave to Helen's image, we need to go back and trace the course of those earlier events. Jessica's reference is to the "sediment" that remained for her from this process--what she made of it at the time it occurred, together with what it became as she recalled it.

In tracing the initial events we find, in fact, Helen making a mesh of intertwined images--"mold," "floating," "tower," "kindergarten blocks," "building"--each interacting with and influencing the others. Looking back at Helen's own story of her experience (as encouraged by Duckworth's questioning), there appear to be four phases in Helen's insightful experience:

1) With the results of Bamberger's intervention, Helen has intimations that there is a mis-fit between those results and the assumptions she was holding up 'til then. Specifically, Helen finds it difficult to reconcile her own and the group's inability to find the next note in the tune with her view that Ricky hadn't "mastered" the task--i.e., his need to start over again each time as he searched for the next note was a "weakness." Helen tells of her sense of something irreconcilable between what had just happened to the group and her own conceptual framework for making sense of Ricky's work. She describes her mind-set in terms of her "mold": "My mold didn't fit when she [Bamberger] did that." And from Lee's comment, ("We certainly went back and played the whole thing from the beginning.") Helen adds, "...what I was thinking all along didn't fit when she said that."
2) There follows a period of simmering confusion. Helen characterizes her feelings during this phase with

another image: "I was somewhere floating--I mean, it wasn't comfortable." To this Helen adds her sense of risk at giving up something without having an alternative: "I had to stay with my old mold and let you go, or I had to put my old mold back there and say, now where do I go from here?"

3) With the help of the group's conversation, Helen then makes a connection with a familiar experience. Through this image as mediator, she begins to catch a glimpse of a possible new view--of "where she could go from here": "I realized when you were talking that music is building--that repetition in music is necessary... well, it's like a tower...and I was thinking of the blocks and I was hearing Lee verbalizing it and I was seeing the kindergarten blocks...then it was starting to get comfortable."

4) With this combination of "verbalizing" and "seeing," Helen finds herself accepting a new view which, interestingly, she sees not as simply discarding the old but somehow transforming it to accommodate the new: "Then it was starting to get comfortable 'cause then I had a--I could take my old mold and put it with the new..."

From Helen's story, then, we get an account of a complex process through which she, almost literally, came to see in a new way. That is, Helen realized that "repetition," which she had initially seen as Ricky's "weakness" ("...he couldn't have mastery of it because if he did he wouldn't have needed that constant.") she later saw as, in fact, "necessary" to his success in tune-building.

How, then, does Jessica see her insight about a "hit" on the computer drum ("...you can't tell how long it is until the next one comes.") as "the same thing as the building blocks?" What, for Jessica is the family resemblance between the two situations? Jessica apparently sees herself as having recognized a familiar phenomenon in her "click" about drum sounds just as Helen had recognized a familiar phenomenon in building which she could also see as appropriate to tunes. But interestingly, Jessica recalls the image as Helen's way of saying what she had recognized: "It tells you something that is always operating in music...I like it because it puts into words what that experience is...it is an easily verbalized something that makes a part of experience that clear." "Building blocks," then, is the name for a moment when you can say clearly and vividly something that only then you know that you know. "It doesn't happen that often!"

But looking back once again at the story Helen told, it seemed to be about a complex, tangled, not "comfortable" process through which she was able to transform her "mold." With this transformation, Ricky's and, indeed, the group's behavior became reasonable, respectable, even useful, rather than a "weakness." Within this process, "building" and "kindergarten blocks" had been images that helped to mediate that change. Now, in the context of the new situation, Jessica and, indeed, Helen, herself, make the image into a stand-alone object that reveals, by "verbalizing," almost on its own. Helen: "It's something you know but you didn't know you knew until somebody verbalized it. Then you say, 'Aha, that's true!'"

But memory, too, is revealing: in the recall, both Jessica and Helen collapse the rather long period of Helen's uncomfortable, "floating" confusion into a single moment of "Aha!" When Jessica makes the connection with "building blocks" (a term Helen never used), the image as she recalls it, becomes both means and instant result--the image, in her memory, carried insight within it. "Building blocks" becomes, in the new situation, a seminal image that had originally, and suddenly, transformed enigma into clarity, unknown into known--first for Helen, when it came to her, then for Jessica when Helen said it. And this sudden clarity is the content of that occasion that both seem to carry over. Now named, it is this experience that became the property of the group--part of its shared history.

Metaphor means, literally, to carry over. As this metaphor passed along through its subsequent recalls, the image named continued to acquire the baggage carried over from each new situation in which someone recalled it. Further, each recaller carried over particular fragments and these gradually joined to form new figures. Over time, all this was somehow absorbed by that entity called "the group" which has its mythology but each individual holds it alone. As phrases, images, "sayings" become materials that "speak," what they are and what they tell is made and re-made each time, along with the telling.

But still, such remembered images may be the stuff, often the invisible stuff, of learning. Giving names to these phenomena made them last, kept them alive to be talked about and used again. Just as the image, "building blocks," seemed to carry insight within it, so this and other pregnant images carried a powerful kind of learning within them. For example, that Helen could live with "floating," that through it she could transform her "mold" into a new one, was something she and, with her, the group, learned that they could do: confusion is the mother of insight. In turn, recognizing what you knew how to do already but only knew that when "it" showed up to illuminate the materials of a new situation, was also a powerful notion--one that carried to the classroom, too.

As Suzanne Langer puts it:

> But between the facts run the threads of unrecorded reality, momentarily recognized, wherever they come to the surface...the bright, twisted threads of symbolic envisagement, imagination, thought--memory and reconstructed memory, belief beyond experience, dream, make-believe, hypothesis, philosophy--the whole creative process of ideation, metaphor, and abstraction that makes human life an adventure in understanding. (Langer, 1942, pp. 236-237, as quoted in Gardner, 1982, p. 50)

We look forward, now, to finding and tracing the course of other critical moments in our continuing efforts to understand the life of the seminar.

## IV TEACHERS AS LEARNERS

### A CASE STUDY ABOUT SOME DEPTHS AND PERPLEXITIES OF ELEMENTARY ARITHMETIC

Here is one extended example of a "reflective conversation with materials." The example is drawn from the second year of the project, in the group of eight "new teachers" - that is, the teachers for whom this was the only year in the project. Three teachers from the first year also participated in this group, in addition to the adjunct teacher (for whom it was also the only year), making twelve teachers altogether.

The group met in alternate weeks, for a total of fifteen sessions. The account which follows describes the ways in which they came to "probe, perturb, make something new of"* their understanding about learning and teaching elementary arithmetic. Mathematics educator Patricia Davidson said of this account, "You worked with mathematics as if it were clay," giving support to our view that coming to new understanding is like making, creating, fashioning something new.

The account also gives support to several other of our views about teaching, learning, and understanding:

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*Overview, p. 7.

that people construct their own understanding for themselves; For anything that matters very much, this construction is long and slow - and risky. Ideas evolve, get more complex, open new questions, get revised and discarded-- and all of this is part of what is involved in developing more adequate knowledge. Confusion and perplexity are advances to the extent that they represent a moving beyond a simple, un-thought through assumption. Learning and knowing are inseparable: learning does not presume an absence of knowing, and knowing does not mean that learning is over. As different people's ideas feed into one another, their differing views need to be taken into account, thus contributing to both greater complexity and greater solidity.

This account begins with the first meeting of the group, and includes segments from ten sessions, including the fifteenth, the last. Most of their arithmetic work is included in the account here, but not all of it. Some discussions which did not so clearly feed into the general developing understanding are not included.

Other themes were being pursued in parallel, most notably, moon-watching, child-watching, and themes from their classrooms. In addition, the second year teachers were doing some music in their own group.

First year teachers, including adjunct teacher:

- Anna
- Deborah
- Heidi
- Karen
- Katharine
- Marya
- Ruth
- Sara
- Vicky

Second year teachers:

- Helen
- Jessica
- Suzanne

# 1. CHIP TRADING - THE BASIC GAME

## Sessions 1 and 2

The chips referred to are, essentially, poker chips, and they are used in a primary school activity aimed at helping children understand about place value. Marion Walter and Ann Manicom, two excellent mathematic educators, have developed some approaches to chip trading activities for teachers which served as the basis of our beginnings here.

To take away the mystification of the chips themselves - and also simply for economy's sake! - we forwent the poker chips and used what we had to hand. For the lowest value, we used S-shaped bits of styrofoam, normally used as a packing material. (We came to refer to them as "squiggles"). Then came straw segments, then rubber bands, and finally wooden hexagons.*

The only basic rule is that, once you have agreed on an exchange rate, then any time that you accumulate that number of any one value, you must trade them in for one of the next higher value. The simplest game is to throw dice and to take squiggles for the number of dots you throw, then trading in the squiggles as necessary. With an exchange rate of three, if you throw a five, you would take five squiggles, and trade three back again for a straw; so you would end with one straw and two squiggles. If you threw a four on your next turn, you would add four squiggles to the two you have, and turn in three squiggles for one straw, and then the remaining three for another; and then, adding those two straws to the one you had from the previous move, you would turn all three in for a rubber band. And so on.

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* This chapter will be made much simpler if the reader furnishes himself or herself with similar materials to move about while reading.

A different game is to start with a hexagon, and subtract the number on the dice. That means you must trade in the hexagon for three rubber bands, trade in one rubber band for three straws, trade in one straw for three squiggles, before you can begin to subtract as many squiggles as the number that you threw.

These two simple games can go on with children for a long time. They usually begin, as do many adults, just as I have described it here - taking the entire total of the throw in individual squiggles, before making a single trade; making each trade by itself, rather than anticipating, for example, that six squiggles could be traded all at once for two straws. Later, some children get very good at such anticipations. Other children stay longer with the insistence on every trade. But they are equally well masters of the game - they equally well know exactly what is going on, what the rules are, how they are doing, what needs doing next. Bit by bit, they notice, for example, that they need one more squiggle to make a straw, and, throwing a two, can simply take a straw and turn in all but one squiggle.

Marion Walter and Ann Manicom have found that it is wise to insist for the first several times through that the teachers also do every step. Partly this is because some teachers, with greater facility with numbers, do several leaps at once, thereby intimidating others who then believe they should do likewise, but find it difficult to do in their heads. For example, in an exchange rate of four, holding two straws and three squiggles, and throwing a six, some teachers simply turn in their straws and two squiggles, while taking a rubber band, figuring that one dot fills what is needed for a third straw, four more make the fourth, which leads to a rubber band, and there is still one dot left over.

There is, however, an even more important reason for insisting that the trades actually be made: Otherwise people can by-pass the whole exchange rate, and never get the sense of the trades. In the above example, a person could say, "I've got eleven, here are six more, that's seventeen; a rubber band is worth sixteen, so I end up with a rubber band and a squiggle." Chip trading is actually more mathematical if the player thinks of the value of a rubber band as so many straws, each of which is worth so many squiggles, rather than jumping from the rubber band directly to its value in squiggles. It is the different-level exchanges which hold the mathematics. This is also a major reason for using small exchange rates - three, four or five: much of the playing time then involves crossing from one level to another, as opposed to simply counting out numbers of squiggles, which takes a great deal of time in an exchange rate of ten.

We began with chip trading at the very first session of the second year group - October 23. The group spent an hour and a half playing the basic game in pairs or threesomes with an exchange rate of three. Some of the time they were adding, and some of the time subtracting. They talked a lot among themselves about how they were doing it, what was easy about it, what was difficult, what was perplexing. "I'm never going to be able to figure this out"; "It's so hard, isn't it"; "I got good rolls that time"; "It didn't ever occur to me that you could logic it out - I'm so used to just doing it." At several times during that working period I stopped them to have a group-wide discussion of questions they were bringing up. Why is it more difficult to subtract than to add? How many squiggles is a rubber band worth? (There was a debate over this question - whether six or nine). What makes this difficult? As they felt they were getting better, what specifically did they think they were getting better at doing? ("I took more risks as I got more comfortable in the group.")

The second session, November 6, we spent about forty-five minutes on chip trading. They were to work with a partner, with an exchange rate of five. At an arbitrary signal they were to combine what each of them had, divide the total evenly between them, and then continue. It soon was clear to them that they had no way of checking on how accurately they were doing that, and I urged them to write down in some way what they had at each stage. Interest developed in how people were writing it down. Here was one way, described by Marya and Suzanne:

| M | S | Together | Each |
|---|---|---|---|
| 4 R | 4 R | 1 H | 4 R |
| 3 St | 3 St | 4 R | 3 St |
| 4 Sq | 1 Sq | 2 St | 2 Sq |
| | | | 1 left |

Deborah and Vicky described another:

"I wrote down what each of us had on a grid:

Deborah

| ○ | o | / | S |
|---|---|---|---|
| | 3 | 4 | 1 |

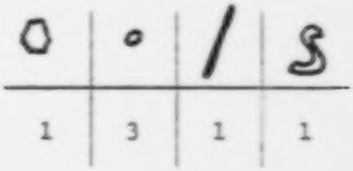
Vicky

| ○ | o | / | S |
|---|---|---|---|
| | 4 | 2 | 1 |

"Then we put them all together and wrote down what there was in the piles before trading in."

| ○ | o | / | S |
|---|---|---|---|
| | 7 | 6 | 1 |

> "Then we did all the trading in and wrote down what was there:"



> "And then we tried to figure out what we should each get, on the paper. Which was very tricky. ... And then we actually did the separating."

Various people were impressed that they had been able to do the dividing on paper. Deborah explained, "Once we did it, then we understood how to do it on the paper."

In the remaining time, everybody tried dividing this collection into two, and then three, and then four. There was some disagreement among the answers, but there was not time left to compare the ways of doing it, and come to understand the disagreements.

Deborah's journal entry about that session was quite characteristic of the feelings about it: "It was neat working in base five last time. The idea of using a hand, five fingers, as a visual and concrete way of seeing base five. Using my hands and Vicky's when needed made it possible to do division as quickly in that base as in base ten. Getting really involved with a material is always exciting. It often brings new depth and understanding to things that you've been working on. What happened during last seminar was that I became more interested in Base five than chip trading."

## 2. FORMULAS, SENTENCES, AND TRIAL BY FIRE

### SESSIONS 3 AND 4

In the hour reserved for chip trading at the third session, I introduced a confusion of my own. I intended it mainly as a small curiosity, perhaps also to serve as an example of a confusion that is easily understandable, and worth noting - and with myself as the person confused - hoping to emphasize that confusion is nothing to hide, on the contrary. I didn't intend it to take up the whole hour, and more. In retrospect, I wish it hadn't. The mathematics got complicated, with not nearly as much milage as the chip trading afforded. However, here is the story of what happened.

At the first chip-trading session, as I was distributing the materials, I had made an assumption that to play the basic game, we would need more squiggles than any other kind of piece, more straws than rubber bands, etc. Later I decided that was not so: we would need just as many of any of them.

As soon as I raised it, Vicky said, "It depends if you have to trade." I said the rule was that you had to trade. "It depends if you're going up or down," she said. We agreed that we were going up.

Karen's first response was that if the base was five, you would only need four squiggles, or else the highest number you could throw. Ruth said, "You don't need more than the number on the dice." Marya explained that you would need four (because that was the highest number you could have in front of you) plus six, because that was the highest number you could throw - ten altogether, then.

Gropingly, they started generating a formula for the number of squiggles a single player would need, given any base[*] and any kind of dice.[+] They let B stand for the base and D stand for the highest number on the dice, and concluded that you would have to have B - 1 (because you might have that many at the beginning of a turn) plus D (in case, already having B - 1, you then threw the highest possible number with the dice.)

Some went on to figure out how many straws would be needed, and this, for one or two participants, was the lowest point in the life of the project. Karen and Vicky came to the conclusion (after a considerable period of time, and many columns of figures on the blackboard) that the number of straws you would need to have available was: $B - 1 + \left(\frac{B - 1 + D}{B}\right)$

Vicky explained the formula this way: "Base minus 1 - that's the most straws that you can have before you. Plus - you're going to get more straws, and the way you find out how many more straws you need is what's in there [in the parenthesis]. You have some ...squiggles, which is also base minus one [that is the greatest number of squiggles you could have at the beginning of a turn] and in addition you throw something on the dice. And you divide all that by the base to find out how many more straws you're going to get....And that's the formula!"

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[*] In this session they started referring to the exchange rate as the "base."

[+] We had only six-sided dice, but they wanted to generalize the problem to hold for other kinds of dice.

We tried this formula again for several of the sets of figures which had generated it - an exchange rate of six and dice of six; an exchange rate of three and dice totalling twelve; an exchange rate of three and some hypothetical dice that could total only eleven. It did always seem to work.

Tentatively, Ruth said, "I understand it, but I don't have the--I'd probably have to trade chips for another hour and write down everything before I could do that." Even then I did not notice that a number of people were quite lost. I proposed instead that everyone try to devise the formula for rubber bands, at home, by essentially doing what Ruth described - trading chips and writing everything down. But I did not respond adequately to her feelings at the time that the whole procedure had raced by her. (This was partly due to my being full of admiration for the clarity with which Vicky had explained the formula.)

Marya managed to get my attention to say, "I've had to put things out in words, whereas you've jumped immediately to the symbols, and I'm sitting here writing sentences, and that just sort of reflects the process that I need to go through,...in order to figure this out. Which is, essentially in teaching kids...it appears to me to be the same kind of thing, where you can't flip to those symbols, which is what I was having...a great deal of difficulty with." Her sentences, which came from the same columns of figures on the blackboard, were the following:

> "If the Die is twice the base, then the rule is base + 1. If the Die is less than one [times the base] or one [times the base], then it's base plus 0."

I asked Ruth if that was what she had been doing. She replied with admirable honesty, and to tension-releasing laughter, "I'm still trading." Marya went on to explain what she found difficult: "I was able to get what's on the board there in the columns, you know, by trading it out, but then I was stuck - 'Well where do I begin, how do I even imagine to come up with a formula?' I can see some similarities of where the numbers become the same, but I wasn't able to flip immediately to- I yet had to do some thinking about that and ...I probably would have done some writing, 'Well, the answer is four if-' and it would have gotten out in sentences."

It was Vicky who pointed out that Marya's sentences really contain more of the relationships than the formula does. She pointed out that the formula simply produces the number of straws that would be necesaary in a given set of circumstances - a given base and a highest possible single throw. Marya's sentences, though, capture a relationship between the answer and the circumstances. "So that what you were trying to do is really harder than this - more information."

Later in the session, Marya talked about when in the proceedings she had let the formula-seekers go their own way, and tried to develop her own sentences. In comparing this session with most classrooms, she said, "It's the wait-time that I don't think is generously given in classroom settings. The minute anyone asks a question, boom, who's ever got their hand up first gets a shot at it, and that's it. It subtracts the space for other people to have other kinds of thoughts."

After hearing Marya, I did try to ask some others who had not been participating much how they might go on from here. Katharine said then, "I understood what they were doing when they did it, but I don't know how they did it. I felt very frustrated, cause I, you know, I felt really stupid that I didn't get it." Ricky Carter, the cameraman, intervened then to point out "the difference between feeling that you are following somebody else's process, and your generating your own - being the generator for one. Katharine, you were saying that you were just following someone else's process, and that was very frustrating."

It was, however, more serious than that. Katharine wrote in her journal that week, and it was lucky she did, because otherwise I would never have known, "I've never felt so 'dumb' in my life...I wanted to fade into the wall. I'm so upset about it I can't even write how I feel." Reading that was probably the lowest moment of my teaching career.

It would have been simple to have avoided that. We did the problem all over again in the next session, people working in their own ways and at their own speeds, using the squiggles and straws as we went, working out many cases, and trying to clarify the relationships between the formula and the sentences, as two ways of answering the same question. Better yet, I should simply have put off the question of a formula when it first arose, and asked anyone who was interested to work on it at home.

In any event, there are two comments to be made. One is that, thanks to Katharine's willingness to write in her journal about her feelings, we were able in a later session to have a discussion about many people's feelings about that evening. This brought the group very close together - though it was too bad that it was at the expense of a trial by fire.

The other comment is that, indeed, in this simplest seeming of materials and possibilities, a question arose which challenged the mathematical tendencies of even those most mathematically inclined among us.

## 3. MATHEMATICAL INSIGHTS

### Sessions 5 and 7

In the fifth session, December 18, we went back to chip trading activities (having come to feel relatively confident about the formulas, although not having understood the relationship between the formulas and Marya's sentences). We had only about half an hour, so instead of engaging the group in an activity of some depth, as I did the following time, I raised a question that was intriguing enough to get people working, dividing and making exchanges, and developing their familiarity with trading phenomena and their various ways of representing them. They were asked to take four rubber bands, three straws, and three squiggles. The question had come up as a real question in the second session: if you wanted to divide this collection evenly between two people, did it make any difference what the exchange rate was? In this session, different pairs interpreted the question differently, did different things, invented different other questions to pursue, and were generally very interested in each other's interpretations and findings. When it became clear that they had interpreted the question differently, they were generally amused that I said that it made no difference to me what they took the question to be as long as it got them working, dividing, making exchanges.

I certainly do not always feel that way about the questions I propose, and when we came back to chip trading in the seventh session, on February 5, my question was quite specific and purposeful. I wanted them to see the powerful mathematics that chip trading could reveal. Otherwise, it would hardly be convincing that the time we had spent on them was more than amusing brain teasers.

I wanted to continue the work of the second session, and I asked them to do what they had been doing then, with one "added wrinkle." I asked them to work in an exchange rate of five. They were to play the basic addition game in pairs taking turns throwing the dice, and trying to get to a hexagon. At some arbitrary time, they were to stop, put together what they had, and divide it evenly. This time, however, before dividing between the two players, they were to divide among some other number of players. When they had done the division, they were to try to figure out a way to check on whether they had done it correctly - whether they still had the same total number. In addition to requiring some way of noting down what they had, this meant that they had either to do all the trades again in reverse, or to figure out some way to multiply.

Maggie Lampert, the observer, noted during this work, "This is, on the face of it, an easy thing to do, not threatening... - they can all get engaged in it without appearing stupid." And after about half an hour: "Everyone/group seems to have constructed their own problems at this point, they all seem into it. When they discover an error, they try to figure out what caused it."

After about 40 minutes, I interrupted their work to ask them all to work on the same problem, so we would have that in common to discuss. I wrote the following on the board:

| 1 | 3 | 1 | 2 | |
|---|---|---|---|---|
| hex | rb | st | sq | (5) |

I asked them to take a clean piece of paper, so it would be easy for them to write down and keep track of what they did, and to divide this, first among two people, and then among five, and check them out each time. After they had all worked on this for ten minutes or so, I asked several pairs to describe what they did when they were dividing it among five people. Here is Marya's account: "We took the one hexagon, broke it down into five rubber bands, and then added it to the other rubber bands, which makes eight. Divided that by five, making it 1 rubber band, remainder three. [Each person gets one rubber band, and there are three left over]. Then took the remainder three rubber bands, broke that down, so that all together with the straws that were already in the straw pile, sixteen. Divided that by five, that's three, remainder one. One straw. Took the remainder of one straw, broke it down into squiggles, and added it, which is seven squiggles---divided that by five and got one, remainder two. So the total answer is 1, 3, 1, remainder 2...1 rubber band, 3 straws, and 1 squiggle, with 2 squiggles left over."

Anna, her partner, then showed how she checked it, as follows:

$$\begin{array}{ccc}
1 & 3 & 1 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
1 & 3 & 1 \\
\underline{1} & \underline{3} & \underline{1} \quad +R2 \\
5 & 15 & 5
\end{array}$$

"So this [the squiggles column] would be 0, this [the straws column] would become 16, and this [the rubber bands] would stay 5."

5 16 0

"And then this [the straws] would change to 1 and that would be 3 [more rubber bands] so that would be 8, and this would stay 0."

8 1 0

"And if I converted that again that would be that."

1 3 1 0 +R2

"And then with the remainder..."

1 3 1 2

I then called on Vicky, whom I had seen to have had an exciting insight. She put the following on the blackboard:

$$\begin{array}{r} 131 \quad R2 \\ 5\overline{)1312} \end{array}$$

"What's happening is, these numbers are written in base 5, and are divided by base 5, so in effect, this [what is being divided] is 1 3 1 2, this [the answer] is 1 3 1 2 - it's like moving a decimal point over, if you were dividing by 10."... "Base 5?" someone asked. "Yeah, well we've been trading 5 for 1, so all these numbers are written in base 5. And in the particular case where we divided by 5, any time we divided by 5 that's what happened, it was as if we moved the decimal point over."

People were mystified by this. That way of writing the problem made no sense, nor did the fact that the numbers were all moved over one place. "So you're dividing 5 into 13," someone remarked. "Could you do it by long division?" someone else asked.

Vicky tried to explain how she had actually done the dividing; it was not unlike what Marya had described; and she ended by saying, "It just works." Karen asked, "But why does it work?" I tried to urge people to try another exchange rate - a suggestion Deborah had made as she witnessed Vicky's discovery. Nobody wanted to take me up on this now, though. They were much too intrigued and tantalized by this mysterious relationship between the numbers. They simply kept looking at it and trying to make sense.

Anna was in fact on the verge of the insight they all were struggling for. She spoke in a very low voice, almost to herself, but those immediately around her were listening. "That 5 should really be a one, oh [one, zero;10]--'cause we're going to be working in base 5." "Aaaah," someone said, with dawning realization. Karen said, slowly, "Excellent. Excellent, five is really ten." The tape records excited raps on the table, and cries of "Aha!" "You got it, you got it!" "You did it!" followed by explanations to themselves and each other, in the same excited tone, all at the same time. "And that's the same with all of them! And that's why it works!" "And then it just looks like any old division." "And that's why it works for the decimals."

"You know what it was?" Vicky summed up. "What you were able to do was put the people in base 5, as well as the stuff."

Remembering the hard lesson I had learned in the third session, I then insisted that everybody take the squiggles, straws, rubber bands and hexagons, and try that same set of numbers, in various different bases - actually doing the trades and dividing into piles. I wanted to be sure they saw why it worked even in base 10 - and not just that the way you write it down is the same. This time they were ready, and worked excitedly and loudly on a variety of different bases.

The most adventurous was a group which tried it with an exchange rate larger than 10. Eleven it was. That seemed to them to be breaking new territory, and to be especially rewarding when it proved, also, to work.

I stopped that work, finally, to talk about a remark I had overheard Heidi make during the initial work, before Vicky's discovery. She had in fact written down in the margin of the paper on which she was working, "divided by the same number as the base." I had heard her say, at that time, "I don't have to check that one, that's easy." She did not see, as Vicky had, that the "answer" was the same set of numbers, moved down one value; she still worked out the dividing by trading and separating into piles. But she somehow knew that she did not have to check those ones by multiplying. When I asked her about it now, she said, "It had to do with the five." She said that each time she checked out the divisions into 5 (this was always working with an exchange rate of 5), they all came out right, she never found she had made a mistake. And she thought, "Well that's ridiculous. If I'm dividing by 5 and it's in 5, I don't need to do that." Heidi wasn't able to say any more about why she thought she didn't need to do that, when she was dividing by the same number as the base, until Ricky Carter had a conversation with her during the break, fortunately overheard by the tape recorder.

Ricky: Did it happen when you were checking?

Heidi: Yes.

Ricky: So you looked at these [straws] and you said, well, there's four of those, I'm going to multiply those by 5. I'll take 5 for each, and so of course that's gonna be-

Heidi: And then I took it right back again, yeah-

Ricky: Yeah, so that's gonna be 4 rubber bands, or whatever.

Heidi: Yeah, right. That's right. You know that it was a step and I was reversing it.

It was not surprising to me that Heidi didn't notice this in the other direction - there is not such a direct sense of taking a step and reversing it. It is surprising to me that, having noticed it in the checking, she had such a hard time saying what she had noticed - why it was "ridiculous" to check. In listening to her conversation with Ricky Carter, I wonder if she did not in fact know all along why it seemed ridiculous, without it occurring to her that that was an interesting level of knowledge to tell about. In responding to her journal, I pointed out to her that that was indeed a mathematical insight she had had - more so than simply to have recognized a repetition of numbers. That comment of mine seemed to have been very significant to her. She referred back to it several times subsequently, when she spent many hours by herself being intrigued with understanding what division was all about.

I was also surprised that she was the only one to see that multiplying by the exchange rate was ridiculously simple. Now, on reflection, I think I understand.

I think it is clear that it was the trading, a sense for the trading, which led to the mathematical insights. All the teams developed perfectly adequate ways to divide by 5 and to check on their division. Some of them worked via the base 10 - translating the total number into a number of squiggles, dividing that number in our own number system, and so on. (Marya and Anna's notation is one example - they counted up to 16 and to 8). Heidi and Vicky, however, thought in terms of the trades. They did not translate into numbers any larger than the exchange rate they were working in. If they wrote down four digits, to represent what they had, they never read them off as thousands and hundreds, but as hexagons and rubber bands. These were the ones who had mathematical insights which turned out to be valuable to all of them. It is a powerful lesson, I think, in how experiences can be masked by pre-learned symbolization.

Vicky, in explaining how she came upon it, referred to having used hands to stand for "how 5 of something is equal to one of something else." She did not carry out the trades with the squiggles, and so on, but thought of hands and fingers. When describing their work in the second session, she and Deborah had commented on how hard it was to do the dividing "on paper", rather than through trading and separating into piles, and now she explained it, along with its relationship to her insight.

> "...I could do it purely with the fingers because of the hands. All I did was, I looked at the figures and I wrote the numbers, because of the hands - that's why I could do that, without doing it. ...When I saw it [that the numbers were moved over 1] it was the same thing, it was part of the same thing. But I was surprised to see it. I didn't just know that it was going to look like that."

Later in the session, after the break, there was more interest in how Vicky had noticed what she had noticed. She explained, "I just want to say that ...the thing with the hands I started the last time we chip traded....It didn't just happen all of a sudden." She then drew on the blackboard a version of what she had previously drawn in her journal.

After a good deal of laughter, this drawing also led to a good deal of thoughtful discussion.

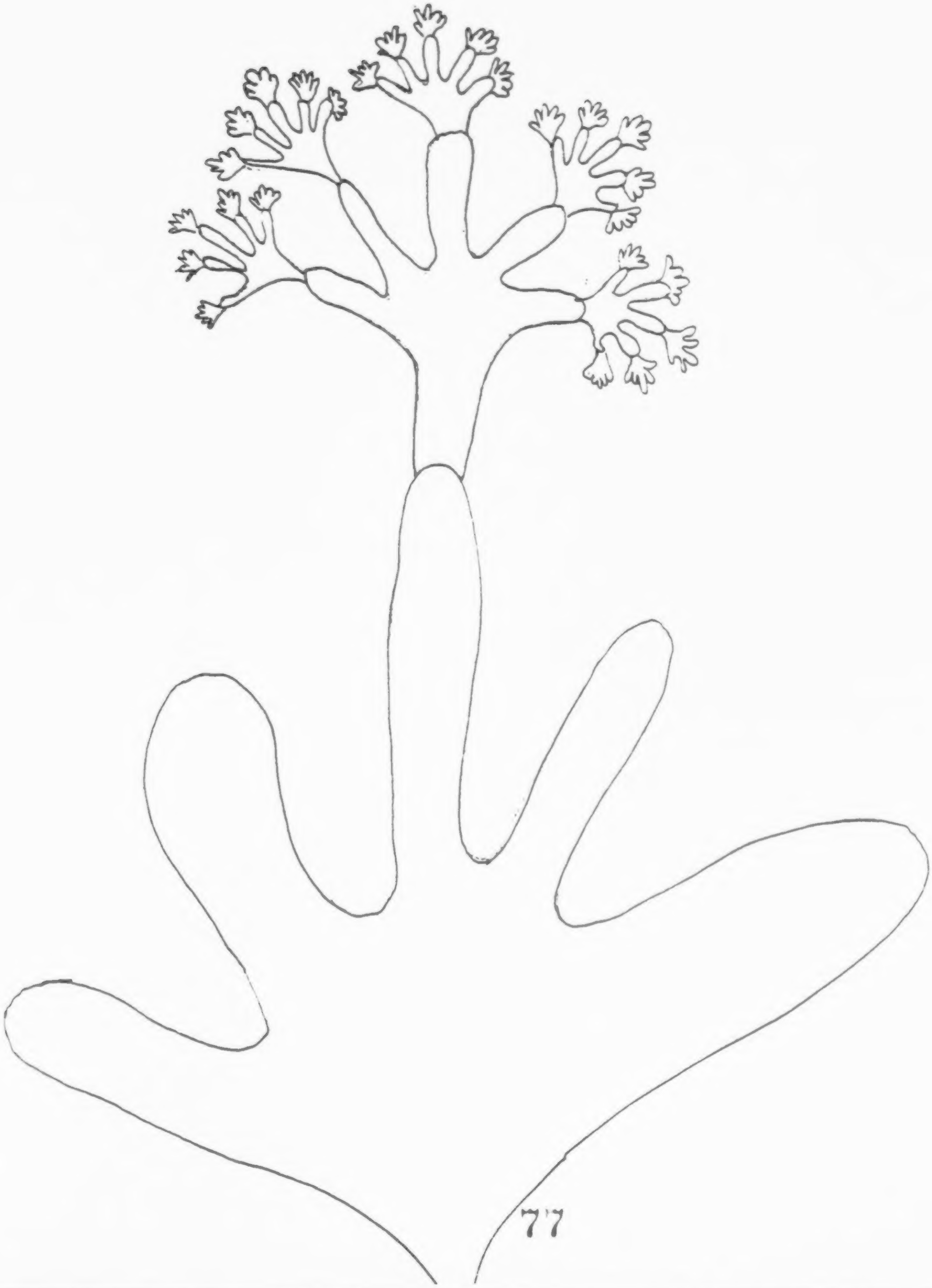
Karen: What does that represent?

Suzanne: Base 5.

Vicky: It's a hand. Each finger has 5 fingers - etcetera.

Ruth: Yes, but what if you have a different base?

Vicky: It's only for base 5.



Karen: Can you show me the units, the next ones up, and the next ones up - which ones are which?

Vicky pointed to the smallest fingers, and each bigger set of fingers in turn, as she said, "Squiggles, straws, rubber bands, and hexagon."

Karen: Five squiggles equals 1 straw, five of the middle-sized is one thumb--

Vicky: One rubber band

Karen: --rubber band, thumb - and five fingers. So if, let's say there were 10 fingers on a hand...this would be a thousand.

Vicky: You know what it is, it's - This is equal to ...

She counted the different levels of fingers, and wrote: $b^4$

Vicky: I think it means that.

Eleanor: Do you think you could explain that?

Vicky: Well, it's like how many times you have to multiply the base times itself.

Eleanor: Maybe you could write that down.

Vicky: We're in base 5, so this [b] would be 5....Five times five times five times five. Twenty-five, a hundred twenty-five...

She now had on the board:

$$5 \times 5 \times 5 \times 5 \qquad \begin{array}{r} 125 \\ \underline{5} \\ 625 \end{array}$$

Eleanor: What is that now, what does that mean, all those numbers?

Vicky: Umm. So I think this would be the number of squiggles [her voice suddenly becomes incredulous] that one hexagon is worth? Is that possible? No. Can't be. [Pause] Yeah-

Karen: Yeah, there's a way, cause if it were tens, hundreds, thousands, ten thousands. But it would be a thousand.

Vicky [Marking off the levels]: It's squiggles, straws, rubber bands, and hexagon. Yeah, I guess it would.

Ruth: You keep squaring each one.

Vicky: Multiplying it by itself.

Ruth: Yeah.

Vicky: That means that in order to get a hexagon you have to roll the dice till you get to 625, which we all did, pretty much...I had no idea it was that many.

Karen: We rolled 625 little dots?! Is that possible?

Eleanor: Did anyone get to a hexagon?

Karen: No. Only by combining.

Vicky: Oh, that's right, only by combining. But still, people must have been getting hundreds.

Many incredulous voices here, and no clarity of ideas.

Karen, business-like: 5 times 5 is 25.

Vicky: That's straws.

Karen: That's the first one.

Vicky: Wait a minute now, it's five, twenty-five, a hundred twenty-five-

Karen: That's right, because in tens-

Sara: But you've got ones.

Sara said this loudly, but nobody took her up on it. I even repeated it: "Sara says, 'But you've got ones'." Still nobody noticed that it was the key to their dilemma.

In every detail of tone this situation was as that described in the Final Report of the Teacher Development Project, 1980, where Timmy simply could not understand why 9 cubes were not enough to complete his tower, and he was impervious to Sandy's explanations. (p. 32)

Suzanne was now proposing that another layer of hands was needed. In the midst of discussing why Suzanne thought that, Vicky said, "Oh yeah, right. Okay. So it's-" And then a sudden change of mind. "Oh, who said 'Because you have ones?'...Right, that's the problem then. I was- Yeah. The ones aren't five to the anything..."

It seemed to me then that we were close to clarity, but I was mistaken. Vicky started to write powers of 5. I interrupted her, saying, "We don't have to get that notation straight, but the point is...how many times do you multiply five by itself to reach the hexagon?" There were different answers. Some said 4, and some said 3, and no one was very clear about explaining why she thought what she thought. I tried to explain why I thought it was 3. I thought I knew exactly how to make it clear to everybody. My definitive explanation elicited not a single sign of interest.

Nobody including me noticed that in fact only three sizes of hand are needed, even though four levels are represented. (When Suzanne had talked of needing to add another layer, people had talked of adding a fifth size.) Vicky came closest in her attempt at a "definitive explanation" - one which succeeded somewhat better than mine.

Vicky: Ok, I got it. This little tiny hand is a straw.

Others: Mm-hum. Yeah.

Vicky: This hand is a rubber band

Others: Oh yeah.

Vicky: And this hand is a hexagon. So you only have to multiply three times.

Ruth: So the hexagon is twenty-five times 5. So where'd you get the 625?

Vicky: I made a mistake.

Ruth: Oh. Okay.

Anna was still not convinced: Because if those fingers are straws, how would you indicate "one" on that drawing?

Vicky: Each little tiny finger this size is a squiggle.

Anna: Okay. And then the little tiny hand as a whole is a straw.

Suzanne still felt some confusion; sorting it out led to some more interesting relationships between what the fingers are and what the hands are: "It's the fingers we're using....they're all connected by the hand, which holds the place....If each finger is a squiggle, then each group of five fingers makes one straw."

Vicky: Mm-hmm. Little one. See, you can't say fingers, because all these things are fingers....of something bigger.

Suzanne: Tiny fingers. Each one of those is a squiggle, right. ...Okay, once you get five of those squiggles, you get a straw, which is another finger on the medium hand.

Vicky: Right. Or, this is a hand, too. But it's also a finger.

[Much general laughter]

Suzanne: Every little hand is worth the next finger...You don't need that big hand, then.

Suzanne still seemed to feel a little perplexed. But exploring her confusion had brought us to our clearest statement of the relationships among the levels of Vicky's drawing. Each handful of five fingers is also a finger, of the next level. Five squiggles are also one straw. You don't need the big hand because the hexagon represented by its finger is also represented by the next-smaller hand.

The excitement of the insights led on to people's accounts of how they felt about the evening. These four give some idea of the significance of this session for the teachers.

Suzanne: The thing I get from, when I'm working with someone else, is trying to understand how they understand something and seeing how, how we meet, or if we're thinking...how we can get to the same answer. Like if I come up with an answer and my partner comes up with another answer and we got to it different ways, I try to figure out how my partner got to that way... The biggest thing isn't getting hooked on 5 and understanding 5, the biggest thing is understanding how someone else is understanding something.- like exercising my understanding of understanding.

Ruth: It's like the moon. I mean, we may look at the moon, and we may see the same thing, like, one night Karen and I compared our moon notes, and they were the same observation but recorded differently. We got the same information but we had recorded it differently. And that's what this is - it's the same information - well, you get different understanding from that - but it's the same information.

Marya: Well I got really excited when Vicky did put that formula on the board, because it related to what I had done...So when she put it on the board - it took me a few minutes to figure out, why is that, what is it, you know, how did she get that answer, and...Anna and I were talking and it came about that I began to understand that what I had done in my way was in fact her numeral representation of it all. So that was very exciting.

Vicky: (To Marya) When you were up there and when you were doing your columns,...it made me realize how, when people develop their own structures for doing things they're gonna look different....This was just one structure, and it happened to be dramatic, but-. I didn't understand what you were doing right away, and then when I did, I realized it was like the same thing I was doing, but it looked different, and that step of just realizing that people are understanding based on their own structure, whatever it is, is easy to forget.

## 4. HEIDI'S QUESTIONS

### Session 11, Part A

We did not take up these problems in the next three sessions, which were taken up with trying to understand the moon, working with children, and discussing classroom incidents.

In the eleventh session, on April 8, I decided to give attention to some people's concerns with teaching long division. A number of them were trying to do that then, and wanted to talk about it. It was, as it always does, posing problems.

Suzanne had written about it in her journal, after a session on music notation, which had taken place in the other group. "In many ways, the varied ways we depicted a particular tune reminded me of my current plight with division. There are many ways of manipulating objects and recording on paper to demonstrate division...the ultimate problem is to get kids to understand the conventional representation of division.

"Here are two conventional methods I know of. I myself prefer the visual place value method (left).

$$\begin{array}{r} 321 \\ 3\overline{)963} \\ -\underline{900} \\ 63 \\ -\underline{60} \\ 3 \\ -\underline{3} \\ 0 \end{array} \quad \text{instead of} \quad \begin{array}{rccc} & 3 & 2 & 1 \\ 3) & \overline{9} & \overline{6} & \overline{3} \\ - & \underline{9} & & \\ & & 6 & \\ & - & \underline{6} & \\ & & & 3 \\ & & - & \underline{3} \\ & & & 0 \end{array}$$

I realize the only difference between the two examples is the presence of zeros in the left example. [Note that there is another difference - the three is 'brought down' earlier.] For me this makes a world of difference...I have decided that the children as well as myself need to do a lot of fooling and moving and working with objects to get a real handle on what it means to divide with quite large numbers..."

I opened this session offering the floor to anyone who had concerns with long division. Katharine explained how she told the children the steps to do, and said that $8\overline{)68}$ was easier than $8\overline{)632}$ , because there was only one step and then there was the remainder; but that $8\overline{)89}$ was easier still, because the 8 goes into the first 8, and you don't have to move over to the next number.

When she had barely begun, Jessica (who teaches younger children, and therefore doesn't teach long division) asked, "What is short division?...What does long division mean - that you have to put all those numbers underneath?" Those who professed to know agreed that that was what was meant by "long division." However, as Katharine proceeded to describe what her kids could do, she said they could do 'one step' (for example, $8\overline{)89}$ ). Jessica said then, "But that's - one step is short division, it's not long division." "Well, no, they can do a remainder," Katharine replied. "I don't consider short division a remainder." Katharine worked out one example, as follows:

$$\begin{array}{r} 11 \\ 8\overline{)89} \\ \underline{8} \\ 09 \\ \underline{8} \\ 1 \end{array}$$

In the midst of it, as she was describing what she told the kids to do, she said, "bring down the 9." Jessica repeated, "Bring down the 9! Watcha do that for, bring it down?" We didn't pursue that question then but went on to the end of the example, and then considered whether it was long division. Katharine said, "Well I don't know, I mean, I just say that short division is when it comes out even." And as an example she put $8\overline{)88}$ .

$$\begin{array}{r} 11 \\ 8\overline{)88} \end{array}$$

"But in fact you can make that into long division," pursued Jessica. "Oh yeah, I have them to that," Katharine said, "Because they're going to have remainders soon and they're going to have to do that....I don't call it short division and long division, *you* said short division and long division." "No, that's right," Jessica acknowledged, "Because if there was long division I figured there must be a short!"

So Katharine's attempt to discuss the kids' difficulties with long division became focussed, instead, on this question of what is long division anyhow, and what kind of division *isn't* long?

Suzanne's attempt to discuss the differences between the two procedures she had written about in her journal got us off into another question. She took Katharine's example $8\overline{)68}$ and as she was in the process of describing how she approaches it, Jessica said, "What you're essentially trying to do is bust this number up, right?" "Break it up into groups of 8," said Suzanne. "No, into 8 groups," said Ruth. Jessica, in her innocence as a first grade teacher, wanted to get it straight. "How many groups of 8 each, or is it break it up into 8 groups?" I should have recognized that this was worth pursuing. My own understanding was that it didn't really matter much which you meant, as long as you realized it could mean either. I didn't notice that for some people, it was one *or* the other - one way of interpreting it was *right*, and the other was wrong. At the very least, I could have taken up Maggie Lampert's suggestion of trying some numbers other than 68 divided by 8, where 8 is *both* the number of groups *and* the number in a group, whichever way you do it. We could easily have had a look at, for example, 8 times 7, where we might be talking about 8 groups of 7 things, or 7 groups of 8 things, and the difference in meaning would have been more evident. It would have saved us a lot of grief later on if I had realized that this was needed and stopped here to have a look at the two meanings. As it was, both Ruth and Suzanne, at some point in their explanations, to Jessica, took the opposite of their own position. Ruth said, "How many groups are there in 65?" - correcting herself to - "How many things are there in each of the 8 groups." Suzanne, on the other hand, said, "If you start with 8 groups, how many are you going to have in each group?" She didn't catch herself.

I simply left this, wrongly taking it to be adequately discussed, and asked Heidi, who had also been writing about division in her journal, for her thoughts. This set us off on many fruitful - if confusing - hours.

"I have one thing to say and that is that I don't understand it, but I think I'm beginning to see some light. It's the only one you do left to right."

Well, what on earth does that mean? Without trying to understand its implications, here, simply, is what she meant.

Take two numbers - say, 7768 and 273. Following the school-taught procedures, if you want to add them,

$$\begin{array}{r} 7768 \\ 273 \\ \hline 8041 \end{array}$$

you fill in the answer from right to left: first the 1, then the 4, then the 0, then the 8. If you want to subtract them,

$$\begin{array}{r} 7768 \\ 273 \\ \hline 7495 \end{array}$$

you fill in the answer from right to left: first the 5, then the 9, then the 4, then the 7. If you want to multiply them, you fill in the answer from right to left:

$$\begin{array}{r} 7768 \\ 273 \\ \hline 23304 \\ 54376\phantom{0} \\ 15536\phantom{00} \\ \hline 2120664 \end{array}$$

first the 4, then the 6, then the 6, then the 0, then the 2, then the 1, then the 2. But if you want to divide them, you fill in the answer from left to right:

$$\begin{array}{r} 28 \\ 273\overline{)7768} \\ \underline{546\phantom{0}} \\ 2308 \\ \underline{2184} \\ 124 \end{array}$$

She had thought she had understood it, until the turn of mind of this seminar led her to question what she thought she understood. "Well, people say it's the opposite of multiplication. Well, subtraction is the opposite of addition and you don't do that. And so I tried doing some backward to see what happened...and the first one I did worked. I can't make it do it again...The obvious answer as somebody said to me is, well, you know, it's just the opposite of multiplication - multiplication you go up, division you go back. Well, that's not- That was always very clear to me until I started looking at it, and it's not clear to me at all. But - I think, what happens is you're borrowing down again - or something akin to that - you're sort of borrowing, I think... You're pushing something over, so maybe it's a kind of borrowing...I just decided I really wanted to work it through methodically."

We spent most of the remaining sessions on that question. Most of us understand something about it now, though most of us still could not say in a sentence or two just how come. I would like to relate some of the ways the question evolved.

Heidi told us of a procedure she had read in a book by Isaac Asimov. "It only works [when you're dividing by] two digits under 20. See, it's limited. But it's a good trick!"

Her example was $18\overline{)4625}$

"I always hate dividing by this kind of number, because I always have to think of it as 20 and estimate it and it's just a pain...I can't do it as 18. I have to do this whole thing in my head of visualizing - I have to visualize these steps. I estimate on 20, because I don't know the 18-times tables. It just takes longer."

If you do Heidi's example as is, it is very hard to do in your head. Most of us would have to write it out, Here's the way I was taught.

(a)

$$\begin{array}{r}
256 \\
18\overline{)4625} \\
\underline{36\phantom{00}} \\
102\phantom{0} \\
\underline{90\phantom{0}} \\
125 \\
\underline{108} \\
17
\end{array} \quad R17$$

Asimov's procedure, however, was to break the 18 into two factors, 2 and 9 (multiply them together and they give 18), and divide the large number first by 2 and then by 9. These two parts most of us can do in our heads, because we know the multiplication tables up to 10. That, at least, seemed to be the idea.

It was obvious to nobody why this might work. In fact, it was clear to very few people where the 9 and the 2 came from. No one asked, explicitly, how you could use that method if you were dividing by 19 instead of by 18 (let alone whether).

However, we tried it, and came out with this:

$$\begin{array}{r}
2\underline{)4625} \\
2312
\end{array} \quad R1$$

(b)

$$\begin{array}{r}
9\underline{)2312} \\
256
\end{array} \quad R8$$

"The remainder's different," said Helen, comparing (b) with (a). "The remainder's different," Heidi repeated. "That's funny because that didn't happen last time" They went over it again, to make sure they hadn't made a mistake with the remainder. Some of them then started looking closely at the figures, to see if there was any sense they could make of them. After a few minutes, Suzanne said, "Oh, I know, wait a minute now,... it has something to do with this remainder. What is that - that 1 - 8 times..." She and others then kept tentatively looking at what might be multiplied or divided. Jessica was getting amused by this seemingly senseless manipulation of figures, and Suzanne finally joined her amusement: "You could just say 9 and 8 is 17 and forget the whole thing." Jessica loved that. "Exactly! And then it would make perfect sense, right?" They both laughed.

Helen wondered what would happen if you divided first by 9 and then by 2. Heidi had originally been thinking of the procedure as a neat trick to increase the scope of division examples you could do in your head, so for her the simplest thing to do was divide in half, and then do the next step. "I think you could start with the 9, but once you've divided it by 2, you've cut it," she said. "I was just curious," said Helen. Then someone suggested using 6 and 3 instead of 9 and 2. This time Heidi said, "You should be able to do it with any of the parts," and the group proceeded to try. The figures were as follows:

| | | |
|---|---|---|
| (c) | $9\overline{)4625}$ | |
| | 513 | R8 |
| | $2\overline{)513}$ | |
| | 256 | R1 |
| (d) | $6\overline{)4625}$ | |
| | 770 | R5 |
| | $3\overline{)770}$ | |
| | 256 | R2 |
| (e) | $3\overline{)4625}$ | |
| | 1541 | R2 |
| | $6\overline{)1541}$ | |
| | 256 | R5 |

As Katharine observed, "So it works except for the remainder." Heidi said, "The remainder is in it, but I don't see why...But [anyway] it's a lot less figures...If you get tired writing figures, it's shorter," thus establishing that it distinctly has some advantages. Jessica, still pursuing her earlier question about what "short division" might be, said, "It's only shorter if you can do it in your head. I had to write it all out because I don't have the multiplication tables down and it writes out the same way...You could do the other one in your head. If you knew the 18 times tables you could do it in your head."

Heidi: "But I don't know the 18 times tables."

Jessica: "Oh I see, I see."

I suggested trying the whole thing with another pair of numbers. "I don't understand the remainder yet," protested Suzanne. "I don't either," I said, "that's why I'm suggesting another. I'm not sure we can get to understand the remainder by brute force." I am not sure what Suzanne or anybody else made of this. What I meant - a principal I go by - is that if ideas have not been presenting themselves for exploration, further looking at these same figures is not likely to give rise to other ideas very quickly. So instead, change it a little, try a variation and then look at what is the same and what is different, and see if other ideas come that way.

Helen suggested trying it "with something other than one," by which she meant a number bigger than 19. She settled on:

$$30\overline{)8976}$$

Various people tried various factors. The "answer" was always 299; the "remainder answer" varied. Here are some results:

```
                 299
               30)8976      R6
(f)              60
                 297
                 270
                  276
                  270
                    6
```

(g)

10)8976
897 R6

3)897
299

(h)

2)8976
4488

15)4488
299 R3

(i)

6)8976
1496

5)1496
299 R1

(j)

5)8976
1795 R1

3)1795
598 R1

2)598
299

(k)

3)8976
2992

5)2992
598 R2

2)598
294

Katharine, who did f, g, and h, pointed out that the remainder came out the same with f and g, but not with h. In the ensuing discussion, Suzanne said, "Maybe understanding the remainder has to do with understanding what you're doing with the things you're dividing," an important thought which nobody picked up on at the time. Its truth came home with great impact in a later session.

Helen was the first to propose some rule that "works some of the time, I think, but...I'm not sure it works all the time...The number you divide by second, plus your last remainder, equals the remainder you get in long division." She pointed to b and i as examples, and also to d, being distracted, I think, by the remainder 5 of the first division. Nobody noticed - or at least nobody commented - that this was the rule Jessica and Suzanne had found so funny a little earlier.

It was easy to find examples where this did not work, and the search went on. Heidi said, "That [this new idea of hers] works out every time. But I don't see why...Take your bottom remainder and multiply it by your top divisor. Well, the only two problems I've done it on have had 2's in the top divisor," [b and h]. "It works with 6 and 5 [i], too," said Helen. "But it doesn't work with 10 and 3 [g]," said Katharine. I then remarked that it looked as if there was something right about 10 and 3, since the 6 is there -- "as if that's the way it should work, and this

[Heidi's] way you had to fiddle around to make it work." There was a good deal of agreement with this. Suzanne was getting bothered: "But that's like coming to the answer and not knowing-." Jessica, laughing again, said, "But that's what most of us do" - laughing sympathetically with Suzanne, in my interpretation - appreciating the outrageousness of shuffling numbers around without understanding what you are doing. But, I think, less bothered than Suzanne, because she felt this would eventually lead to some understanding. Helen took the thought a step further. "We can put ourselves in the place of our kids. They're just trying to 'do it that way', with no idea what they're doing or why they're doing it...Now you know how they feel."

Heidi went on looking for a rule: "If there's no remainder the first time, there's no problem, if there's only a remainder the second time." Suzanne: "But you still don't understand where it came from." Heidi mocked herself in answering, "No, but if I started going into what I don't understand in math...!" But then her tone changed to serious: "No, I don't understand...yet." She is, after all, the one who really set out to understand how long division works.

I chose to leave this now, since nobody had a particular idea of how she wanted to proceed. I expected some of them would be intrigued enough to work on it themselves, and we could come back to it again as a group. Before leaving it, I asked Suzanne to say again what she had said about needing to understand what you were doing. "You have to know what you're doing with the things - with the numb--with what you're using." "Yeah," I said, "In order not to find just... I mean in order to understand why these worked." Several others chimed in their agreement with this thought now, and Helen said, "We didn't know what the remainder was, by doing it the long way," by which she meant that now at least they know that they don't know. Confusion is headway.

Suzanne said, "I need stuff to move." "Okay, what kind of stuff," I asked. She hesitated. "Chips. I relate real well to chips."

For now I tabled Heidi's question. "First question from Heidi is, 'How come in dividing only you go from left to right?' Second question from Heidi is, 'How come this dividing by factors in turn works except for remainders and how come it doesn't work for remainders?'" Suzanne: "Maybe it does work for remainders." Eleanor: "And maybe it does, and how can we make it so? So those are question 1 and question 2 parts a, b, and c, to be considered."

That part of the exploration had taken about an hour. I now took up another set of questions about division.

## 5. THAN'S PROCEDURE AND SUZANNE'S QUESTION

### Session 11, Part B

Suzanne had referred to a way of doing division that had come up in her class, so she now told us about that. The problem was $8\overline{)291}$.

"Now this is the way I interpreted what he wrote down... He had this mishmash of numbers all over his paper... It got the idea from what he had on his paper, how I could straighten it out in my mind, and how it looked to me, and he agreed with me at the end that this was the way it was."

She wrote these numbers:

$8\overline{)291}$

8
16
32
64
128
256

"He knew he was close here" (at 256).

Then she pointed to each in turn, starting with the 16, saying, "So now he counted and he said he had 2, and that was 4, and that was 8, and that was 16 groups, and this is 32 groups." Later, she wrote these numbers down:

| | |
|---|---|
| 8 | 1 |
| 16 | 2 |
| 32 | 4 |
| 64 | 8 |
| 128 | 16 |
| 256 | 32 |

"And then he had to add some more groups to get to 291, and he picked out a number that looked like it would fit." She added 32 to the bottom of the first column, along with a total of 288.

$$\begin{array}{r} 8\\ 16\\ 32\\ 64\\ 128\\ 256\\ \underline{32}\\ 288 \end{array} \qquad \begin{array}{r} 32\\ \underline{4}\\ 36 \end{array} \quad \text{R3}$$

Then she said, "This was 2, 4, 8, 16, 32, [groups of 8, to reach 256] and another 4 groups, that's 36, plus you needed 3 more to get to 291." Jessica: ("Hm! Pretty snazzy!").

In the midst of a number of appreciative exclamations, Katharine asked, "How did you get 288? You didn't add those up [the whole column], did you?" Suzanne: "Yeah." Katharine: "Oh, just the last 2." Suzanne: "Just the last two, 'cause he's adding another set of groups."

Suzanne then mimicked Than asking, "'Why can't I do it this way?' I said, 'You can. You understand it'." Helen said, "He probably knows it better than the rest of them."

"Well, today during the lesson he said to one of the kids, 'It's easy, it's just D, M, S'...That meant divide, multiply, and subtract, you just keep doing that over and over again...He was referring to the conventional way.

Somehow he got from his procedure, and I said, 'Yes, you do know how to divide.' And...he had a chance in between times to fool around with some materials and do some problems on his own and today when I was explaining the conventional way to the kids he said to one of them, 'Huh, it's easy.'" "Do you know what he meant by that," I asked, "Do you know the connection between that and this [his way]?" Suzanne: "I think it might- In my mind it was that he understood what it was to divide, that he was moving numbers around, and that once he could visualize his own way, of figuring out the problem and understanding it and getting a hold on it, that he could see my way of doing it and understanding it, too. As long as I explained it to him, the way he explained his to me."

This last remark - almost an afterthought - is one of the most indicative of the respect for children's thinking that developed in this group of teachers - a reciprocity: you do your best to explain your way to me, I'll do my best to explain my way to you; and we'll each do our best to understand the other's way, too.

All of us were impressed at Suzanne's having been able to do this - to take a "mishmash of numbers," realize that there might be some sense to them, work to find the sense, and appreciate the child for a way that was different from the way she had been trying to teach.

A question remained, though. Suzanne believed that Than's having developed his own way was an important element in Than's coming to understand her conventional way: "Once he could visualize his own way, of figuring out the problem...he could see my way of doing it and understand it, too." But it did not seem to follow that the same might hold true of other kids - that encouraging them to invent their own ways might be an alternative approach of trying to explain to them her way. In her journal, as quoted above, she had written, "The ultimate problem is to get the kids to understand the conventional representation of division." It now seems to me possible that Suzanne was toying here with the idea that, though "ultimate", the problem might be unnecessary; that the difficulties arise from trying to teach the conventional way, and if only one did not try to do that, long division would be much simpler. This did not seem to be clear to her, though, and she continued to feel it her responsibility to teach as the books said to teach. The issue came up explicitly in the final session, and is discussed later.

Helen was impressed by how much Than understood, "He's got the grouping, you know, doing it that way... He's really got the idea that dividing is all grouping, you know, and he's - using it."

Heidi then, still concerned with what division really is, said, "In fact, wasn't he multiplying until he got to where he needed to go?"

Suzanne: "Um, multiplying? Well, yeah. I mean multiplying is just repeated adding." Heidi: "Yeah. But- Is there a distinction between that and dividing? Given a number, he started at 8 and multiplied up - well, did he multiply? No he added - up till he got to match the number...rather than...actually dividing it."

There followed a question about where the second 32 came from, and where the 36 came from. It would have been a good idea here - as it would have been earlier in the matter of 8 groups versus 8 in a group - to change the numbers. The fact that 8 was getting doubled in both columns was a source of confusion. The two 32's, in fact, had no particular relation to one another. If I had suggested switching to a number divided by 7, for example, things would probably have been easier to think about.

7)255

| | | | |
|---|---|---|---|
| 7 | 1 | | |
| 14 | 2 | | |
| 28 | 4 | | |
| 56 | 8 | | |
| 112 | 16 | | |
| 224 | 32 | 36 | R3 |
| +28 | +4 | | |
| 256 | 36 | | |

And of course, it would have been a good idea to take some other example, anyway, just to be sure we knew what was happening, and to make sure it happened in other cases. This was another instance, though, where I got carried away with my own interest in the problem, and did not pay enough attention to what everybody else was making of it.

In any event, after answering the questions about the 32's and the 36, Suzanne returned to Heidi's question. "I'm interested in what you were saying about whether that's dividing or multiplying." "Or repeated adding," said Heidi. "I can see it as- I think I can see it as dividing. He's just adding the groups of 8," said Suzanne. "Until he gets up to the number," said Heidi. "It obviously works," she continued, "But I'm unclear that that's dividing." Helen said, very haltingly, "I think it's more dividing than- I mean, what we say is how many groups of 8 are inside 291, and he's already got that concept. I mean he knows he has to figure out how many groups of this number 8 are inside that big thing, and he - he realizes that - to get that group you just gotta keep, I mean, to get that group what you do is you add, you're adding groups. To divide, you're taking those groups away, and you say, 'Well, how many groups do I have now?'"

Heidi said, "You're just adding them up, rather than taking them away." If I had been alert at the time, I think I would have amended what Heidi said, to say, "You're just adding them up as you're taking them away," to see whether that would have seemed helpful to anybody. Helen responded to Heidi by saying, "I think it really shows that he's got division." But that wasn't a response to Heidi's concern. Heidi was not concerned, right here, with what Than understood. She was concerned with, What is division?

She described how she thought of division by using unifix cubes - "little plastic blocks that you can stick together." "Start with that problem. Now there's at least a couple of ways you can do it. Start with 291 and start taking off blocks of 8. Now that to me is dividing, cause it's removing. Or what I see him doing is starting with 8 and moving up, matching each time 'till- to see where he's at." Suzanne: "It's almost the exact same thing, because if you're taking off a group and dividing- separating it from the rest, you're left with a whole bunch still..." "Right," said Heidi, "...but he had to stop at some point and figure out... how much more to go, whereas if he's taken 291 unifix and started pulling them off in groups of 8, he just would go until he didn't have a group left." After several more minutes of discussion, she explained, "I'm not criticizing his method at all, just trying to analyze whether in fact that is what we call division or whether he is coming at his answer by using other processes." This reminded Jessica of her own question, what is long division. "I mean is it the adding up the groups at the end...or is it all this subtraction in the middle, or is it..." She then went on to what division means to her. "When I think about that 291 and 8, I think, 'OK, can I easily put 8 into 200. No. So I'm going to put 8 into 20 tens. Right? Find out how many tens I've got left, I mean that's what I'd do in my own head to simplify it for myself." There was laughter here, as few seemed to have followed that description. "All I'm saying is that I don't know whether division is the adding up of the groups of 8 at the end, or if it is the- is somehow, the busting up process. But the busting up process it seems to me varies depending on what is easiest for you...For him it's busting it up...efficiently, really."

Suzanne, continuing with Heidi's question, said "I understand it as division." She indicated two collections of things on the blackboard. "If this means a group of 8, right, there's 8 things there. This is a group of 8 and that's a group of 8. Now he pictures in his head that he's got two groups of those things. So now, isn't that the same thing as dividing? He's separating them into groups?" "But then he has to keep adding them up and seeing where he's at," said Heidi. Jessica responded, "Don't you have to count the groups of 8 unifix cubes you get off every time? You gotta add 'em up." Heidi: "But you can count 'em, 1, 2.." There were many objections to this. "But that's even more primitive than what he's done," said Helen - again focusing on what he understood, rather than on what it means to divide. "I'm not arguing that what he did is much more sophisticated. What I'm trying to do is go back and settle in my own head what's division," Heidi explained again. Helen insisted, "If we consider that dividing [what Suzanne just did], then we should consider this dividing [Than's procedure], because this is just a more sophisticated efficient method of doing that."

The question did not get further resolved. It seemed clear that long division entails both "busting up", and some how keeping track of the "busting up" - which requires counting, adding, and/or multiplication. Nobody quite mentioned either the role of the subtraction, to compare how close you are to the original number; nor the role of guessing, to begin with.

It seems to me now that it would have been helpful at this point to ask for ideas about when it is that we need/want to divide. What are some questions for which this is a way to get an answer? This would turn Heidi's question of "what is division" slightly differently. I do not know whether she would have accepted it as a way to approach her question. In her journal she subsequently wrote, "Does division imply a process or an answer?"

Helen wanted to say yet more about how good Than's understanding was. "When I look at that 291, if I was to do that, I don't look at that as a whole number, I would go 8 into 29, and 29 is 29, you know?

```
   3
8)291
  24
   51
   etc.
```

And then I'd go down and I'd go 8 into 51, and that's it. And then 51 is 51 and it's never 291 to me."

"What if someone doesn't understand 29?" Suzanne asked. "What if they start saying, 'Well whaddaya mean 29, that's not 29, that's 291!' That's how I got into the 0, keeping my place value there, because I found it very difficult to explain to them why - where you get 36 from in that problem.

```
     2
18)4625
   36
```

'Where's 36 come from?' And then I'd say, 'Well, it's not really 36...it's more than 36.'"

Jessica then said, "That also solves the left to right problem." "Yeah, it does," said Helen. "And that's what I do in my head, because I can't..." Eager to know how people were thinking about 'the left to right problem', I asked Jessica to explain. "Because you're dealing-- You're dealing-- You're not-- You're starting with the whole-- You can start with 291 units, once, right? Or you can start with...[There was a long pause here, with nobody interrupting - everyone, I think, hoping Jessica would manage to shed some light on this funny question.] And then I move over to 10's usually, and I usually stop at 10's, because that's easier for me to-." She stopped. She certainly had an idea of what she wanted to say, but her words did not after all seem to make conncections for anyone else.

Instead, Helen went on to comment about how confusing she found Suzanne's approach of writing all the 0's, while appreciating that in fact it might be more understandable. She then said that she in fact does division in a way very similar to Than's. "I often do that, group it and double it...I wish my kids would do that...I didn't do that for a very long time, until I finally understood how to divide, and then I used to go laboring through 18 times 2, 18 times 3, writing it all out on your paper...until I finally came to one that was close to 46, and then I'd write it down, and then I'd subtract, and I'd go through that whole really painful experience. And what he's got there- As an adult that's how I divide now, rounding it off, doubling it, and it makes it very easy, into nice little separate packages, and then how many separate packages do I have, and Bingo I have an answer."

That made three people now - Heidi, Helen and Jessica - who described the way they "really" did division - not ways that resemble the conventional ways that are taught in school.

Heidi, during this description, was thinking about what Jessica had said, and clearly was ready to say something herself. When I asked her what was on her mind, she said, "Well, I'm getting there. Jessica was saying the right to left...It's really this."

$$\begin{array}{r} 6 \\ 50 \\ \underline{200} \\ 18\overline{)4625} \end{array}$$

"Right," said Suzanne. "Sure," said Helen. "Exactly," said Jessica. "And that," said Heidi, "would seem to make a lot of sense!"

Now unfortunately, I had in mind during this whole discussion a procedure which I had worked out and wanted to contribute to the attempts to understand long division. This insight of Heidi's was reminiscent of some features of my procedure. Instead, then, of following this up, instead of making sure to engage some of the teachers who had been silent for some time, and exploring Heidi's insight - by trying it with other numbers, by comparing its steps with Than's steps - I wanted to show them my procedure! It was a great mistake. What I did, by being too keen to introduce it, was to miss the chance to take off from Heidi's fine insight, which was close to everybody right then. As I was preoccupied within myself about how to present my idea, the teachers discussed a few ways that various textbooks propose teaching long division - including one that was the equivalent of Heidi's earlier idea with unifix cubes - subtracting 8, then

8 again, then 8 again, and so on - as a way of trying to help children understand what was going on as they did the division procedure.

I then - reluctantly, because I was still eager to get on with my own idea - gave Suzanne the floor to ask a question which clearly was important to her. But again, to my dismay now, I did not take the time to develop it.

"Um - What's the purpose of teaching division?" Suzanne asked. "When we're teaching them in the primary grades, what are you interested in, them getting the answer, or them understanding what division is? They're going to be dividing until they graduate from high school...My feeling is if they can understand, why they divide and how they divide, then that's my purpose in teaching them, that they understand it, not that they come out with the right answer."

That, of course, was a major point of the enterprise. But I didn't seize upon it and draw a discussion from it. I asked if anybody had anything to say, but it was clear to them that I would prefer to go on to what I had in my mind. I could have asked someone - Heidi, Katharine, Sara - what she thought of what Suzanne said, and I think the discussion would have been significant. To the extent that I did have any reason, for going on with my agenda, it was that there were still things about long division that we hadn't yet talked about, which I intended my analysis to raise - and then, when they understood, it would be a good time to have the discussion about teaching for understanding. Of course, you never understand everything, and you always understand something - so, if the question comes up in such a natural, pressing way, then is the time to pursue it. In any event, at this point I went on with my demonstration.

"I want to put something on the board and see what you make of it...Like Than, who had those numbers all over the paper, and Suzanne had to figure out what they were. So I had these numbers all over this paper, and I thought it might be interesting to see if you could figure out what my numbers are. It's a huge great big problem, just because I thought that's what long division was. I didn't realize it was $8\overline{)89}$ ...Mine is: I've got 75,381 something-or-others,... I was thinking about our chip-trading, dividing it back into piles, and here's what I did." I then put these figures on the board.

75,381 into 221 piles.

300/pile [Here I said, "That means 300 per pile, OK?"]

300/pile = $\begin{array}{r} 221 \\ \times 300 \\ \hline 66300 \end{array}$

$\begin{array}{r} 75381 \\ 66300 \\ \hline 9081 \end{array}$

40/pile = $\begin{array}{r} 8840 \\ 66300 \\ \hline 75140 \end{array}$

1/pile = $\begin{array}{r} 221 \\ 75,140 \\ \hline 75,361 \end{array}$

341/pile with
21 left over.

When I stopped there was silence for a long time. The first remark was, "I'm getting a headache." After another long silence, Heidi ventured, "Is this, 'If there were 300 to a pile,'-" and Jessica continued, "then you'd end up using that many. You'd end up using 66,300 and then you take the 66300 out of the 75381 and you get that 9081 left over. And then, if you have 40 for a pile-." She started slowing down, and other supporting voices can be heard. "Wait a minute now, yeah...then you'd get that 8840." A few people took some time to make sure they followed, and Jessica continued. "And so then you add it to the 66300 and you get that [75,140], and then you got--" She stopped, and there was some discussion now which revealed that Ruth had been thinking of my figures as three separate problems.

Jessica went on, "Then you add it together [getting 75140] and see how much you've got left over - I mean, how much you've used up- and this-." Helen interrupted here to point out that I had made a mistake in the remainder, and when Jessica got back on track, having referred to her notebook where she had done some figuring, she finished: "And then you've got this pile of 221, you add that up and this is how much you've got left and then you've got...21 left over." Perhaps it was due to the interruption, but note that Jessica refers here to a pile of 221, whereas the representation is really for a single item distributed into each of 221 piles. Nobody picked up on this switch - and I didn't notice it myself.

Heidi still wanted to check her understanding and try out some alternatives. "Where- . I was trying to figure out if- . Well you just sort of...guessed, OK, let's take 40, right?" I agreed. "What would happen if you took the 8840...and then subtracted that from the 9081 that you had left over?" "You did that in your head, didn't you?" asked Helen. After some thought I said, "I didn't do that. That's interesting...Instead I did this, and I made this comparison [75381 - 75140] and it comes out to the same number - 241. I didn't subtract these [9081 - 8840]; instead I subtracted those [75381 - 75140]." Heidi, revealing the pervasive assumption that there is only one right way, said, "Yeah, because that's all you had to subtract," as if her suggestion must have been wrong, because that was not how I had done it. "Well, I could have subtracted this, just like you said," I replied. "and they gave the same number, and I happened to go about it the other way."

There was a little further discussion, up to a point where Heidi said, "This is exhausting." "Exhausting?" "Yeah, well I felt very anxious until I could kind of talk about what was going on...because it was too many figures and I couldn't figure out what was going on. At first. And then it was OK. Because I could clarify it." I didn't ask who else had felt or still felt anxious. I am now sure that some others did - judging from their silence, as I go back over the tapes, and from Maggie Lampert's notes. I should have given them a chance to say so, and to find out what they made of this. I have learned since, in teaching basic math to university students, that big numbers, for many people, tend to have no meaning. They are just things you apply rules to. Playing around with the rules, then, is exceedingly distressing - there is nothing left to hold on to.

It would have been useful, at least, to go back to Heidi's earlier insight, and fill in some of the numbers it would have given rise to, in order to compare them with mine. Heidi's insight would have given this:

```
          1
         40
        300
    ______
221)75,381
    66,300
    ------
     9,081
     8,840
     -----
       241
       221
       ---
        20
```

When I asked whether consideration of my procedure contributed anything to the discussion, there was a long silence. Sara said, hesitantly, "I see it as a way of estimating." Nobody drew a connection to Heidi's insight, nor recognized that the numbers produced during this procedure were related to those that would be produced in the internal parts of conventional long division.

Helen responded in a more general way, referring to the way that both my procedure and Than's procedure retain the meaning of the numbers: "It seems to me that that's more efficient than...the way that I do it...Because...you have this idea that you have so many there that you have to group. And you take the whole thing and you handle the whole thing and you get guesses and you come close, and you get it. In my way I take 75 or 753 [instead of 75381] and I go from there. So I haven't really got the sense of the whole thing...I can get the right answer. But I'm not so sure that I have the whole sense of-. I'm not sure if I didn't have that recipe I could figure it out. The recipe that I learned."

All of this - all that is described in these two chapters - had happened in the first half of one session. It was clear by now that everybody was ready for a break. I was uncomfortable leaving this without offering people the occasion to draw more conncections. And, since my procedure had not had the magical effect I had thought it would, we were in no better position than we had been before, to take up Suzanne's question about the reasons for teaching long division.

Two journal entries are significant in appreciating this session, and its different meaning to different participants. Heidi pursued the mathematics, over many pages, and over many weeks. Just after the session, she wrote:

> "Doing the following problem (in class) has helped me tremendously.

```
       6
      50
     200
18)4625
   3600
   1020
    900
    125
    108
     17
```

Breaking it into parts made sense. I have done this with addition with kids, but it never occurred to me that it applies to everything."

She tried Than's proceudre and my procedure with different problems, and after these efforts she wrote:

"How does all this (Eleanor's) relate to Long Div.

```
17)431
```

Step 1 17)43 , which is really 17)430

One estimates and multiplies and then subtracts from the whole.

Step 2

```
   20
17)43(0)
   34
    90
```

90 left from 430

Step 3

```
    25
17)431
  -34
    91  (90+1)
   -85
     6
```

Take that 90 and add the 1. Repeat, breaking into 'piles' of 17.

Isn't this a short cut (less writing) to Than and Eleanor's way?"

She tried breaking the number you divide <u>into</u>, into its place value parts (400 + 30 + 1), tried breaking the number you divide by into place value parts (10 + 7), realized the difference between breaking 16 into 8 times 2 and breaking 17 into 10 plus 7. Her pages were peppered with questions and answers to herself. Here is one example: "No, I don't see. I 'see' but don't understand, but I will. Next day - I'm not sure. It's very long, trading into ones, but I want to. I'm stuck. Help!"

She was, of course, not stuck, and kept on exploring.

Suzanne, on the other hand, wrote the following in her journal a few days later: "We were all discussing division and the methods of recording what it means. It almost seemed as if people were very 'into' the manipulation of numbers regardless of the meaning they (the numbers) rendered. I found it very difficult to be satisfied with that trend, and tried to bring up the aspect of what do the numbers mean, what sense can be made of them? This seemed to be on the minds of almost no one. It was dropped. Perhaps this was a too philosophical or 'too scary' a question. Maybe the group was relating to it (division) differently than I. I was coming from a day of trying to make division make sense, a week of frustrated kids moving numbers around and not really grasping an understanding of it all, a unit [music notation] whose purpose was to make the recording make sense.

"I felt a sense of satisfaction from the above seminar mainly because I had been made aware of some growth in myself. In that sense and in others I feel that mixing new blood is healthy and profitable."

"Tackling similar classroom related topics may be just as stimulating for all groups. It [long division] is an area that is extremely difficult to teach with meaning and some of the most bizarre misconceptions are exhibited by children (a challenge and a mystery). I see the 'music pictures' as a very similar exercise although not as easily identifiable with the classroom as the division. With the music we got more into what the symbols mean, what do we see, what does the symbol reveal - how does someone else interpret it. - Ah ha! Maybe that's why I felt as I did about division."

The last class of this group - four sessions later - did start to address children's understanding of school-taught arithmetic procedures, and we shall get to that later in this account.

# 6. MARYA'S FORMULA

## SESSION 14

With the approach of the end of the year, realizing that only two sessions remained, and that we had made many openings that remained unexplored, I wrote up four of our problems, and encouraged people to work on one or more of them for themselves. (See end of this chapter.) Of these, the one that we then took up together in the fourteenth and second last session (May 27) was the 'remainder' question, from Heidi.

Marya had decided first to try to find some formula that worked, since she thought that if she knew that much, she could then find a way to understand <u>why</u> it worked.

"I came around it back end first. I decided, well there's got to be some formula for...how it all works out. So I started looking around with what I had as my remainders and multiplying them,...going along with the idea that there are formulas that govern the math world, that some multiplying, adding and dividing would get me something."

The problem I had proposed was 651 divided by 18. It turned out this way:

```
     36 R3
 18)651
    54
    111
    108
      3
```

```
[A]   2)651
        325     R1      [B]

      9)326
         36     R1      [C]
```

Marya's rule, which worked on this problem though it was not a problem she had derived it from, was: "Of your second division problem you take the remainder [C], and multiply it times your first divisor [A], add it to the remainder [of the first division, B]." In this example, that meant 1 times 2, plus 1 - which equals 3, the same remainder you get when you simply divide 18 into 651.

While Marya had so far given her attention to finding a rule that worked, Vicky had developed some understanding of these relationships. "What I did was I just- I knew that this remainder [C] was a remainder for each group - so I would just multiply that 2 [A] by this 1 [C]. What happened to me was that I realized that the reason you could divide by 2 and then by 9 was because first you would divide each of those smaller piles into 9 groups to find out what the answer would be - if you wanted 18 equal groups. And so I just in my mind had the very strong image that there were going to be two groups, and that whatever happened when you divided it by 9 was happening twice."

Marya had not gone that far, had not yet figured out how her formula "worked", so Vicky pointed out the similarity on Marya's own problem, 55 divided by 12.

$$\begin{array}{r} 4 \\ 12\overline{)55} \\ \underline{48} \\ 7 \end{array} \qquad R7$$

[A] $\begin{array}{r} 6\underline{)55} \\ 9 \end{array}$ R1 [B]

$\begin{array}{r} 2\underline{)9} \\ 4 \end{array}$ R1 [C]

"The 6 in yours [A] is the same as the 2 here, because it's the first divisor, and the remainder 1 [C], is the same as this remainder 1 because it's the result of the second division. In other words, first you divided them into 6 groups and you got this 1 left over [B]. Well, that's just 1; and then each of those 6 groups had this remainder [C], when you divided that in helf, so you multiplied this [the second remainder, 1] by that [the 6] for each of the groups."

Now a diligent reader might follow this if she/he put her/his mind to it. But most of us did not really grasp Vicky's explanation until much later. Marya had an inkling, but she was still not certain she understood why her formula worked - what it was doing. Referring to the second remainder 1, she said, "It's got 6 in there. I don't know if that explains it - but this isn't really just 1." She went on to say, "The vocabulary that you use and the way you line things up can lead you to either - not see, or - see the problem in different ways." She proposed that with blocks it would be much easier to see why her formula worked. It turned out that it was not easy at all. It was astonishingly complex, and took us the rest of the three-hour session.

We started with very small numbers - 30 divided by 8 - and tried to clarify our understanding by showing the relationships in "the way you line things up." Maggie Lampert noted at the end of this session, "I was terrifically bored and antsy during the division discussion, - feeling that it would be cleared up if everyone would just slow down and look. Eleanor's role?" The trouble was, much of the time I couldn't understand what was happening, as we tried to show with blocks how the Heidi/Asimov procedure works. After trying to make sense of the tape of this session, I now see most of the relationships, and I would like to try to present their development here. I think this is the extreme example I have run across of the complexity contained within something that seems simple.

After half an hour of work in pairs, Marya tried again to show by using blocks what she now understood. "OK... breaking down 30 so that you have 7 groups with 4 in each, with a remainder of 2...

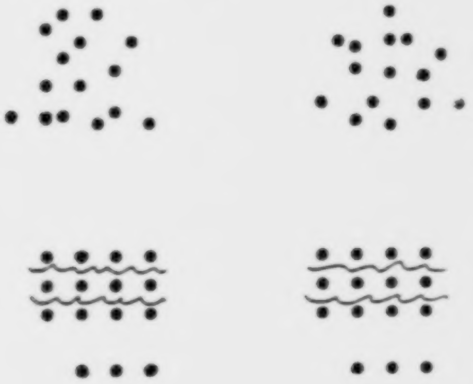
```
●●●●
●●●●
●●●●
●●●●        ●●
●●●●
●●●●
●●●●
```

She had 7 groups with 4 objects each. Then when she went to sub-divide one of those groups, dividing into groups of two objects, she got lost- as, I must say, did I.

If you make groups of 2, within one group, you get 2 blocks per sub-group. How does that show 30 divided by 8? Some of the teachers saw how to get out of their bewilderment, but attention did not immediately turn to what they were trying to tell us.

Instead we turned to Sara, who had stuck closer to the original way of doing this procedure, namely, first divide it in half, so you are working with a smaller number. "It's easier if you divide it into two first, so you've got halves. But then I don't think you can divide only one half into four. You've got to divide the whole thing."

In the first step, Sara did two things differently from Marya: For one thing, she started with the 2 rather than the 4; and for another, in dividing by 2 she made two groups, rather than making groups that each had two blocks. Marya, rather than making four groups, had made groups of four blocks each. Everybody was aware of the first difference - starting with the 2 rather than with the 4. But not everybody was aware of the second difference. This was the distinction that we failed to examine thoroughly enough when it first came up in Session 11, and now it was about to get us into great perplexity. One person did say, "She's counting how many in each group, and you are counting how many groups!", and a few others agreed that that was worth noticing. Nobody suggested what she should change her approach, though, and she continued,



while saying, "So that goes into 15, 1, 2, 3 [as she took three groups of 4] remainder 3; 1, 2, 3 remainder 3. That's 6 with a remainder of 6." And then she corrected herself, "I mean, 3 with a remainder of 6." That troubled a number of people, for a number of reasons. For one thing, how do you know that you now add the remainders, but not the 'answer'? Others were bothered that she had done something different from what they had expected, and, they thought, different from what she had intended.

Suzanne: "First you divided it into two groups and then you said you divided each group into four groups" (she actually hadn't said she would do that) "but you didn't, you divided each group into three groups."

Sara: "Did I?" She looked again at what she had done.

"No, I am dividing it in fours - into groups of four, 'cause we're now doing the 4. I divided it into two, now I am going to divide into four."

Suzanne: "You're dividing a different way now, aren't you?"

Sara: Well, how do you divide? You take a four, you take the fours away, don't you?"

Suzanne: "When you divided by 2, you did something different."

Vicky: "You separated them into two groups; now you are putting four in each group."

Sara: "No - Umm. Isn't that the same thing?"

Suzanne: "No."

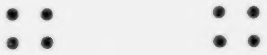
Sara: "To divide it in half I divided the whole group in 2. Now I'm dividing each group into fours - by 4."

Note that she corrected herself here - from "into fours" to "by 4." It is as if she had some inkling that they weren't the same.

It is probably worth noting other ways of expressing the relationship, which have come up here. We heard "in 4," "by 4," "into 4," "into 4's," "by 4's." It would be hard to say which way of dividing is referred to by each of those, though I tend to think that when "4's" is used it is likely to mean "groups of 4." In general I think the various forms are of no help in making clear which procedure one has in mind. It is also worth noting that we sometimes talk of dividing "4 into" 30, still meaning the same thing. Sara at one point, talked of dividing in halves (and one might, though I don't think anyone did, talk of dividing in half). And someone, in an effort to point out the parallel between her first division and what she might now do, referred to dividing into quarters. In fact, of course, you can't divide a group of fifteen blocks into quarters. You can divide them into four equal groups, but because of the remainder the groups aren't quarters - 3 isn't a quarter of 15.

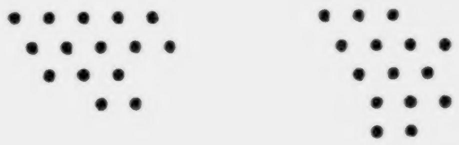
In response to Sara's asking, "Isn't that the same thing?" they took "a simple number" to show again what the two different meanings were. They were pleased and satisfied to stop and show for the first time, "That can mean, 'how many groups of two are there in eight,'

or 'divide eight into two groups.'"

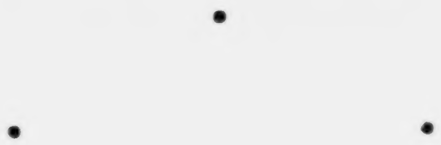


It was at this point - two hours into working on this problem, and a month after the question first came up, that one of the teachers said, "The language is different - but the answer's the same." In fact it is not only the language that is different: the relationships among the blocks are different. But it is true that despite this difference, the answer is the same. Sara was right that they are "the same thing" in that they lead to the same answer. This kept having an impact all through the 2 years' work - the answer is not everything. All kinds of meanings are contained in the work done before one gets to an answer. Eight divided by 4 (into 4, by 4's, into 4's, 4 divided into 8) can mean, with objects, two very different things, although the 'answer' - the number that comes out as an answer - is the same.

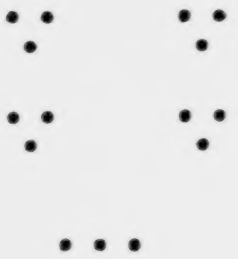
And note that even in dividing a straight-forward number problem -- no word problem here - you still have to know "2 what? two groups, or two in a group?" Vicky said this well after Suzanne, finally, did pursue Sara's approach through to the end. "I divided the 30 into two groups," Suzanne said.



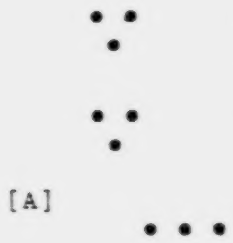
"Now I am going to take each group and divide it into four groups." She put a block in each of four different places, as markers and beginnings of her four groups.

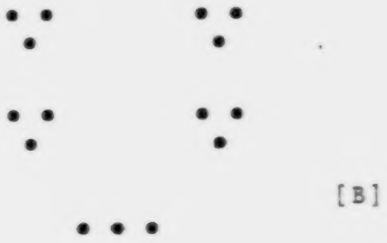


"I'm going to keep filling my four groups 'till I run out." She did that, until there were 3 in each group. "Now I can't put any more in the groups without leaving one group without one."



Then she did the same with the other group of 15. "So, now I have four groups and there's three in each group and there's three left over here [A], and I have four groups and there's three in each group and there's three left over here [B]. So in each group I have 3, and my total remainder is 6."





Someone then asked, "What's your answer to the problem?" Suzanne replied, "Three remainder 6." There was still a little hesitation until someone specified "...in each group." This is where Vicky said, "If you divide into that number of groups, your answer is going to be the number in each group. If you divide so that each group has that number, your answer will be the number of groups." In some way, Sara's first and second steps were "the same thing" (they got to the same number) and in some way they weren't (the "answer" was the same but the question was different).

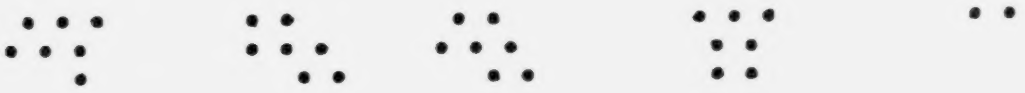
For some poeple, after all this time, this was the first time it was clear that there were these two possible meanings to the enterprise. Others had seen that clearly already, and had other questions. One we referred to earlier - why, in Sara's second step, does one get the right number only if one looks at what happens in one of the groups of 15 (three groups) whereas in Suzanne's second step, you get the right answer if you look at the total, as well (three in a group). As far as the remainder goes, however, there is not that distinction. To get the remainder, you have to total up all the separate ones, both in Sara's second step and in Suzanne's second step.

Other people were intrigued that, counter to their expectations, you could switch from one meaning of division to the other in the middle of Heidi's procedure, and it would still work. That was in fact what Sara had done.

This still bothered Karen, though. She said two things about this. "If you switch halfway, you don't know which you are talking about" by which she meant, "where do you look for the answer, in the number of groups, or in the number of blocks per group?" And she also said, "Somewhere in this problem I want to see the 8."

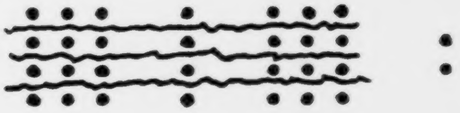
"What 8?" a couple of people asked. "You are dividing by 8 originally," said Karen. "Oh." "And even if you come out with the right answer, in this picture [Sara's] there's no 8." Which seems to be true. In Suzanne's picture, there are 8 groups, when you look at the total. Where's the 8 in Sara's picture? Karen thought there wasn't one, and that was why switching in the middle bothered her.

To explore this further, they returned to Marya's problem, of dividing first by 4. They began, however, making four groups of 7, rather than, as Marya had done, seven groups of 4. Karen started by making four piles, with 2 left over.



Then she said she would "divide each pile again in half," (Sara is heard to say, "by 2." I think that correction was intended to keep it clear that Karen was dealing with the two factors of 8.)

"I have one left over, I'll put it in the remainder pile." But someone else suggested she leave it where it was, "so we can see where it came from." Her picture ended up like this:



Ruth said, "That's very neat" and several connected it with the numbers that were still on the blackboard. "This

$$\begin{array}{r} 7 \text{ R2} \\ 4\overline{)30} \\ \underline{28} \\ 2 \end{array}$$

is the part where she ended up with four piles of 7 and a remainder of 2 - those two little white ones. And this

$$\begin{array}{r} 3 \text{ R1} \\ 2\overline{)7} \\ \underline{6} \\ 1 \end{array}$$

is the part where out of each group of 7 she got two piles with 3 in each and one left over - in each."

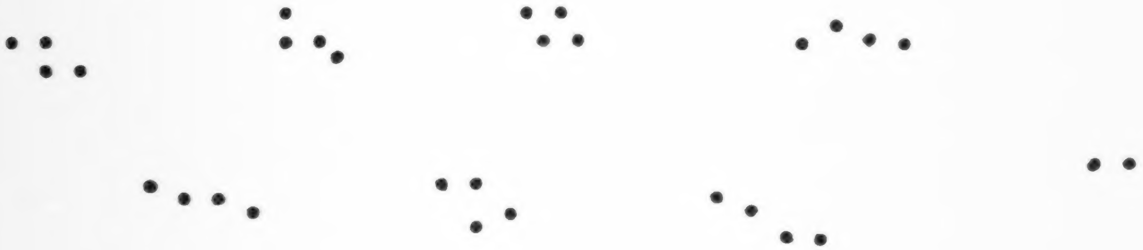
Not sure whether this was clear to everyone, I asked, "There you've got R1 and R2...How come this really means R6?" My hunch was right. Heidi said, "That doesn't work for me either. I can't see it."

Vicky tried to explain. "Because this is happening four times."

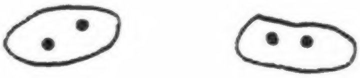
$$\begin{array}{l} 2\underline{)\ 7} \\ \quad\ 3 \text{ R1} \end{array}$$

Marya chimed in agreement here, "four times." And went on to say, harking back to her formula of the beginning of the session - "So that's why the 1 times 4 works [C times A]." Vicky, agreeing, said, "Because this represents what you're doing in the little groups. You have four little groups."

I wondered now whether we could tackle Marya's "picture." I knew that *I* still did not know why hers hadn't worked. She began it again - seven piles with 4 in each, and 2 left over.

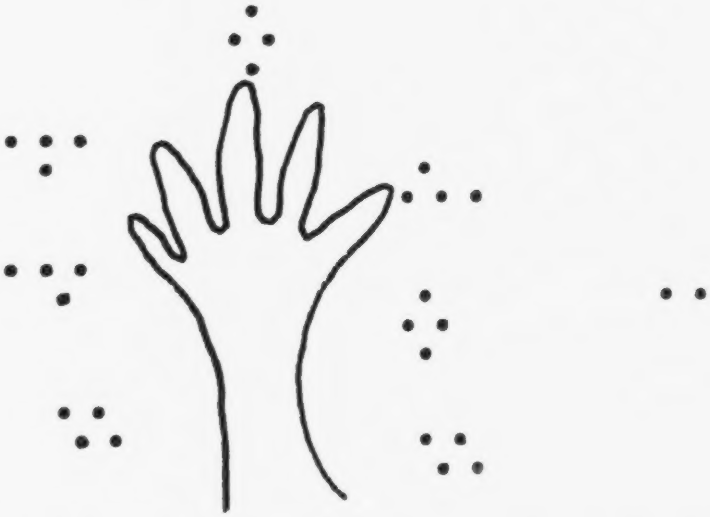


"Now I am going to put 2 in each pile," and she began this way:



"There I go again."
Karen explained, "Each of those piles of four blocks is now a unit." "Why?" I asked - playing no devil's advocate this time. I really didn't know. "You can see that it works if it is, but why is it?" Vicky said, "Because the answer, now, after the first step, is the number of piles. The 7 means the number of piles. You could take each pile, throw it away and get a little piece of paper that said "pile" on it and you could use that one piece of paper now and manipulate that and get your answer."

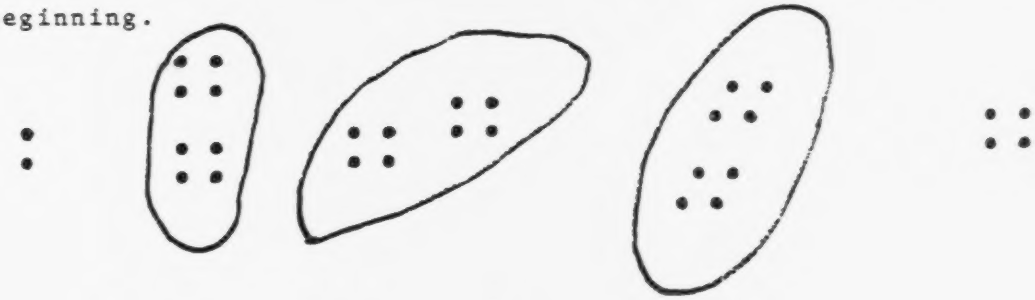
That almost made sense, though Marya was not clear how to proceed until Jessica laid Marya's arm down among the piles so it looked like this:



Marya laughed. "There's the remainder 6, OK, [4 at the top and 2 at the side] and I've got two piles [left and right]." Various voices agreed with this or protested it, and Marya and Jessica both said, "I've got two groups of three piles - with 4 in each pile. ...and a remainder of 6." Jessica added, "Remainder 1 of that type, right? [the one left-over group of 4], but then you have to translate it into 4. You could write down 'remainder 2 plus 1 group of 4-in-a-group' ...and that would also work. That's Marya's formula! Yeah! Is that your formula?"

Suzanne was still unhappy with this. "That came out right. I don't think it should have."

Vicky then did the second kind of division, from Marya's beginning.



"Now you can see the 8."

Karen then came up with what she called "this wonderful thought," which took us right back to the first class of chip trading.

"You said, 'How did you know that that one pile of 4 was really little 'ones,' and you said something that reminded me of chip trading, and I said, 'OK, if I had thirty squiggles and I was doing base 8 and every eight squiggles I trade in for a straw' - so I would get three straws and six squiggles left over. It was like - 'oooh, where did that come from?' I couldn't believe it; it doesn't make sense to me at all."

A number of us had a fragile hold on this comparison. Mine was too fragile to enable me to pursue it right away.

Suzanne then proceeded to say, "Still, how do you know what to add to get the remainder?" and Jessica tried to explain now. We didn't ever get to know whether Jessica's explanation made any difference for Suzanne, because the time was up; but she ended up referring to Karen's link-up with chip-trading. "Four goes into 30 7 times, alright, Suzanne? And you've got one group of that - one pile, I like it better-- of those--, of four in each. So you have to tran-, trā-, trade it in! [laughs at her use of chip-trading terminology] for four ones; trade in that remainder 1 for four ones - take it up, add it to the 2 - you get 6 left over."

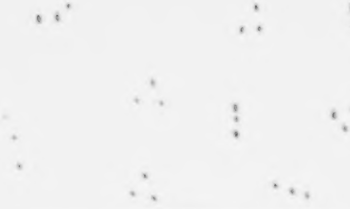
The remainders have place values!

Vicky went on to explain what the multiplying means in the two different forms of division: "It's either one pile, in which case there are 4 in it, so each pile has 4, [Marya's way] or it was just a remainder of 1 which you got each four times, [Karen's way]."

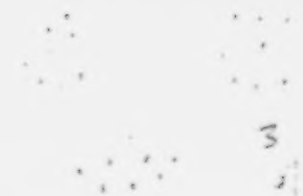
So Marya's puzzling formula turned out to be understandable as a form of place value: whatever the second remainder is "stands for" some number times that number - for one of the two reasons which Vicky summarized.

We still left feeling quite unsure of all that we had done. Much was, in fact, still left to do - in spite of having spent three hours on it. But it gave us an excellent opening for the next and last meeting.

As an exercise for myself, as I attempted to understand this session from the tapes, I made a set of representations of 24 ÷ 8, which I include here for the reader's interest (see next page). Note that this is a simplified problem--there are are no remainders! Represented are: 24 divided by 8, with the two possible meanings; 24 divided first by 2 and then by 4, each time with the two possible meanings; and 24 divided first by 4 and then by 2, again with the two possible meanings each time.



8 piles of
3 things



3 piles
of 8 Things

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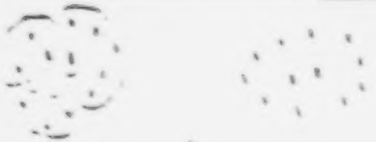
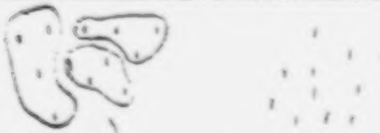


2 piles of
2 Things

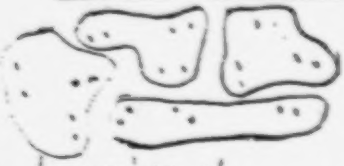
4 groups of 3 things in each pile



2 piles
12 things

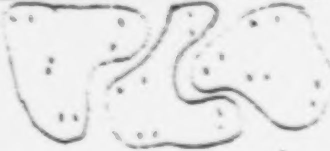
3 groups of 4 things in each pile

---



12 piles of
2 Things

4 bunches of
3 piles of things



12 piles
of 2 things

3 bunches of
4 piles things

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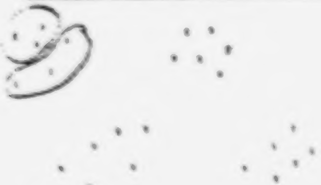
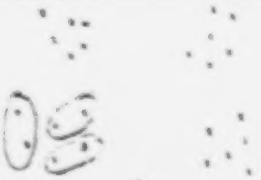


4 piles
of 6 Things

2 groups of 3 things in each pile



4 piles
of 6 things

3 groups of 2 Things in each pile

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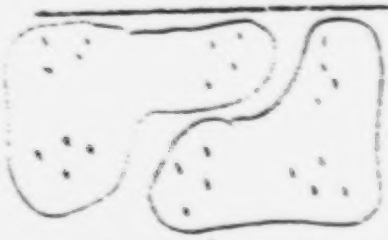
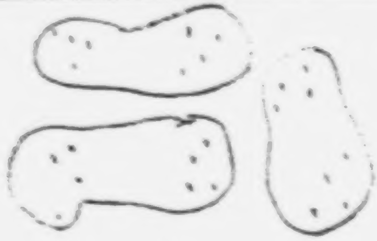


6 piles
of 4 Things

2 bunches of
3 piles of Things



6 piles
of 4
Things

3 bunches of
2 piles of things

24 ÷ 8 = 3 where is the 3?

where is the 8?

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The following five pages are copies of the problems I wrote out as reminders to the teachers of still unresolved questions. The first is the one which was pursued in this session.

**Heidi's procedure :**

$$18\,\overline{)651} \qquad\qquad\qquad \begin{array}{r} 36\\ 18\,\overline{)651}\\ \underline{54\phantom{1}}\\ 111\\ \underline{108}\\ 3 \end{array}$$

$$\left.\begin{array}{rl} 2\,\underline{)651} & \\ 9\,\underline{)325} & \text{R.}1 \\ 36 & \text{R.}1 \end{array}\right\}\;?\longleftarrow\;\Big\{\text{R. } 3$$

How come it works approximately?

How come the remainders do what they do?

18 can also be 9 × 2, 2 × 3 × 3, 6 × 3, 3 × 2 × 3, 3 × 6, 3 × 3 × 2. Here's a couple more of them:

$$\begin{array}{rl} 2\,\underline{)651} & \\ 3\,\underline{)325} & \text{R.}1 \\ 3\,\underline{)108} & \text{R.}1 \\ 36 & \end{array} \qquad\qquad \begin{array}{rl} 9\,\underline{)651} & \\ 2\,\underline{)\ 72} & \text{R.}3 \\ 36 & \end{array}$$

To try to understand it, try out some examples using squiggles and other chips.

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Than's procedure:

8 )291

8 (1×8)
16 (2×8)
32 (4×8)
64 (8×8)
128 (16×8)
256 (32×8)
⋮

.... Remember? Starting with 8, he was doubling each time, approaching 291 as close as he could get. There are thirty-two 8's in 256 and then four more 8's gets him up to 288; so the closest he can get is thirty-six eights, and there's 3 left over.

---

Here's another example:

6 )237

6 (1×6)
12 (2×6)
24 (4×6)
48 (8×6)
⋮

How does Than's procedure work in this case?

---

How about this case:

17 )431

17 (1×17)
34 (2×17)
68 (4×17)
136 (8×17)
272 (16×17)
⋮

HOW ABOUT ANY OTHER CASE YOU MAKE UP?

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## Eleanor's procedure:

75,782 squiggles to be divided into 204 piles.
How many squiggles per pile?

- Well, I'll guess you could put at least 300 per pile

$$\begin{array}{r} 204 \\ \times\ 300 \\ \hline 61200 \end{array}$$

That uses up 61,200, and leaves 14,582

- So now I'll put 70 more per pile

$$\begin{array}{r} 204 \\ \times\ 70 \\ \hline 14280 \end{array}$$

That uses up a total of 75,480, and leaves 302

- So now I'll put 1 more per pile

204

So I can put 300+70+1
and I have 98 left over.

---

What does this have to do with long division?

Do something like it in some other examples
(e.g, 43151 squiggles to be divided into 170 piles)
(or 431 squiggles to be divided into 14 piles)

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## Vicky's Formula and Marya's sentence:

Do you remember the question?

- First we wondered, for one person to do chip trading for a while, how many squiggles have to be available? We (Karen!) soon realized that it depends on the exchange-rate and on how high is the highest number you might throw on the dice.

  Vicky (or was it Karen – or Anna?) used b to stand for the exchange rate and D to stand for the highest number you can throw at once. Then you'd have to have b-1 (because ~~that's~~ you might have that many at the beginning of a turn) plus D (in case, already having b-1, you then threw the highest number possible with the dice.)

- Then came the next question: how many straws would have to be available?

  We tried a few examples, and came up with the figures on the next page:

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| base (exchange rate) | highest number possible on one throw of Dice | | number of straws needed |
|---|---|---|---|
| 2 | 6 | → | 4 |
| 3 | 6 | → | 4 |
| 4 | 6 | → | 5 |
| 5 | 6 | → | 6 |
| 6 | 6 | → | 6 |
| 7 | 6 | → | 6 |
| 3 | 12 | → | 6 |

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- From these figures, Vicky got this formula:

$$\text{no. of straws needed} = b - 1 + \frac{(b - 1 + D)}{b}$$

- Marya, on the other hand, got these sentences:
  - If the Die is twice the base, then the rule is base + 1
  - If the Die is less than one [times the or one [times the base], then it's base +

WHAT'S THE RELATIONSHIP, IF ANY, BETWEEN VICKY'S FORMULA AND MARYA'S SENTENCES?

## 7. CREATIVE ADDITION

At the outset of the last session, I put on the blackboard something I had found in my notes. I no longer know how it had gotten into my notes - whether I had seen it in somebody else's during the previous session, or whether I had overheard the comment while the figures were on the blackboard, or what- I didn't know.

> "
> $$\begin{array}{rl} 3 & \\ 8\overline{)30} & \text{R6} \end{array}$$
>
> $$\begin{array}{rl} 4\underline{)30} & \\ 2\underline{)\ \ 7} & \text{R2} \\ 3 & \text{R1} \end{array}$$
>
> The moral of this is no one understands anything."

It elicited remarks such as, "Very appropriate," "Very true," "It felt that way." Ruth offered a variant: "The question for us is does one ever understand anything?" We did not dwell on it.

I went on, then, to read a transcript of part of a tape I had just been listening to. It was a tape from many weeks before, and it had kept running, inadvertently, during the supper break, recording this conversation anonymously. Heidi was nonetheless easily recognizable, since she had since then brought her question before the group. Vicky confessed participation.

Heidi: Did you say a couple of times ago that you didn't understand division? Well it just occurred to me that I don't understand it; I haven't looked; now I'm gonna look. It just hit me that it is the only one you do left to right.

**Vicky:** Say that again?

**Heidi:** Division is the only one that you do left to right.

**Vicky:** Oh--huh!

**Heidi:** And it just hit me--that I don't understand that.

**Vicky:** You know the funny thing I've just noticed--I can't subtract any more. I can't subtract with re-grouping any more in my check book.

**Heidi:** What do you do?

**Vicky:** I just can never figure out whether it should be in 9's, whether I should be adding 9's or whether I should be adding 10's, and I always have to... I don't trust anything unless I check it by adding up.

**Heidi:** Yeah.

**Vicky:** And I just really don't know--I always knew. I never had any trouble, and now I...

**Heidi:** The more you teach, the more you think about it, the harder it really is.

We asked Vicky a little bit more about her subtraction difficulties. "It was just this period of a few weeks that I really couldn't do it. I just could not do it." I asked, "Was it due to the fact that you were playing around with how to do it?" Vicky: "I think, so. I think I couldn't ...rely on whatever little plan of action I had, which was very mechanical. Like maybe the place values were just really unreal or something."

I asked Heidi what she had meant by the closing remark. "I was taught a lot of that stuff by rote, and in teaching kids, I look at it differently, and it's not as easy." She talked about the gropings in her journal. "It's just on and on, and I certainly understand what you mean about your check book."

A number of others acknowledged experiencing similar phenomena, and Marya was finding that some of her spelling automatisms were becoming shaky. Vicky then tried to describe what it was that she could not do for a while: "What I didn't have for a while was when I was borrowing-, when I borrowed and when I didn't - especially if I had to borrow for several places in a row..."

I asked whether they thought this confusion was unhealthy. At first the question got lost, but Heidi made a point of coming back to it. "To answer your question, I don't think it's unhealthy. I think it's how we learned it in the first place that's unhealthy." After a brief discussion of the relationships between learning number facts and understanding what you are doing, I re-introduced a story which had been discussed in the twelfth session, April 29, when the teachers were each asked to tell about an incident in their classes where children did or said something that they found puzzling, and what they had done about it. Vicky's story was about Jonathan, a second grader who had "a real real sophisticated understanding of numbers and how they relate." He was doing an addition example, like this,

$$\begin{array}{r} 1548 \\ \underline{238} \\ 1786 \end{array}$$

"He got the right answer. But he never used any carrying marks at all...When I asked how he got this answer,...I think what he said was, 'Seventeen hundred thirty-eight, and forty-eight, and I just thought about it and I got seventeen hundred eighty-six.'" With a gesture she showed that first he read off the fifteen from the first line and the two hundred thirty-eight from the second line, and then went back to add on the 48 from the first line. Vicky explained how he added on the forty-eight: "I think what he does is, here he would go thirty-eight, forty-eight, fifty-eight, sixty-eight, seventy-eight, and add the eight on." She went on to say, "I decided that I thought it was important for him to be able to use the carrying...where you carry the one, in case he had really long strings of numbers...I _think_ that you need to be able to do that - I'm not positive about that." She then described trying to teach him about carrying marks. He finally learned to put them in the right places, but, said Vicky, "He clearly did not understand what the point of all the little ones was..." "He didn't need it," said Anna. "He didn't need it, but he didn't really understand what the point was...We weren't jiving, as far as what we were giving each other."

In bringing up this story again here, I asked, "If he does this in this way...why would one think it might be important for him to use carrying marks? Why not go with this? That's the general question. It relates to Than's procedure, of long division, where...I had a feeling that some people might think that that procedure's all very well - 'he sure does understand long division, doesn't he - Okay now let's teach him the way to do it'...'He sure does understand adding, doesn't he - OK now let's teach him the way to do it.' ...I wonder what he would have ended up doing if he had tried to extend that procedure in his own way."

Vicky said, "He got the wrong answer...'Cause that's exactly what happened. This kid was saying 'I want to add up millions of big numbers'...He wrote them himself, he said, 'This is what I got when I added them up.' And then I added them up and I said, 'I got bla bla bla, can you understand why I did?'...and that [the carrying marks] helped him to get the right answer.'"

I then added some figures to the example:

$$\begin{array}{r} 1548 \\ 238 \\ 381 \\ \underline{1682} \end{array}$$

and asked each person to invent "some way she might add these up that's more like what he did here than like [conventional carrying]...Just try to invent some way to add those up that's not the conventional way, essentially."

They each worked for a few minutes, and then I asked Sara to put her way on the blackboard. She put:

$$\begin{array}{r} 2000 \\ 1600 \\ 230 \\ \underline{19} \\ 3849 \end{array}$$

She said, "I just went the opposite way, as if you were reading. I added up thousands, tens and ones." I asked her where the sixteen hundred came from. "If you add all the hundreds together you get sixteen hundred." Someone said, "I like that." Someone else: "That makes sense to me." "Division left to right makes sense when you do it that way, too," said Heidi. "Do you realize that's addition left to right?" I asked. "Long addition," someone said, to general agreement. But Jessica said, "Nothing about that takes longer." Several people explained at once why it is long addition. "You write out all the zeros for each place value," was one remark. Suzanne, meanwhile, said, "Want to have a race, Jessica?" While conversation went on, Jessica kept ruminating, and said, "I don't think that's true, Heidi, that it's longer....I think that's the rationale for there being a 'right way,' it's that for some reason it's shorter, but I don't think that's true that it's necessarily shorter." "It looks long," said Karen. "Oh I know, I know," said Jessica, "Remember we went through that when all you guys were talking about long division and I didn't know whether that's what you meant, whether that was what it was supposed to do: look long." "That's 17 figures, and that's-", said Heidi. "I know," said Jessica, but I assume you're talking about time." Then their attention came back to the way that Marya had just put on the blackboard:

$$\begin{array}{r} 1500 \\ \underline{500} \\ 2000 \\ \underline{1600} \\ 3600 \\ 230 \\ \underline{19} \\ 3869 \end{array}$$

Suzanne pointed out that Marya's was the same as Sara's after adding the 1600, "But what she did was, she added the thousands and the hundreds together, in the same frame. She took two steps to add the thousands and the hundreds." Several of them explained that the fifteen hundred came from the 1548, and the 500 came from the two next lines. I did not notice, and nobody else mentioned, that Marya's and Sara's 1600's did not come from the same place. Marya's was from the 1682; Sara's was the total of the numbers in the hundreds column. Similarly, the 2000 in the middle of Marya's is not the same as the 2000 in Sara's. Marya's is 15 plus 2 plus 3; Sara's is the sum of the two ones in the thousands column. It is sheer coincidence that each procedure has a 2000 and a 1600.

Katharine was next to show what she had done. "I added up each column."

| | | | |
|---|---|---|---|
| 1 | 5 | 4 | 8 |
| | 2 | 3 | 8 |
| | 3 | 8 | 1 |
| 1 | 6 | 8 | 2 |
| 2 | 16 | 23 | 19 |

Then from the column totals she got the "answer" in this way:

2 ‖ 16 ‖ 23 ‖ 19
3   8   4   9

"And then I said, well, that [the ones] will be 9, carry 1 so that's 24 [the 23 becomes 24] so that would be 4, 24 so you'd take the 2 [from 24] over there so that would be 18 [hundreds], and take the 1 over there [to add to the 2 thousands]." There were many exclamations of appreciation, and Heidi groaned! "That's incredible. She just moved it to the number next and that takes care of place value."

Vicky said, "That's the same thing as...Sara's." This at first was surprising. Vicky explained herself, "You say the 2 means 2 thousands, the 16 is 16 hundreds, the 23 is 23 tens, and there's 19 ones. "It's a short form of Sara's," Suzanne observed. "She just didn't write all the zeros," said Jessica. "Yeah," said Vicky. "But because it's in that place, then you know what they are."

A private conversation between Suzanne and Jessica can be heard on the tape, as the group proceeds to the next one:

Suzanne: I like that.
Jessica: Why?
Suzanne: Why do I like it?
Jessica: You're the one who makes the kids put the zeros down.
Suzanne: Yeah, but-. Yeah, I know. (laughter) But that's a good way to get-. You know when you're doing chip trading, and...you get all these things in

one column. Then they have to do their trading - which is exactly what she did. She had all that stuff in columns, in the places. She had, like, 16 hundreds in one place, 23 tens in one place. All of that's illegal, she had to make it legal, so she only has one number - in each -"

Jessica: I don't understand a word of what you just said, Suzanne.

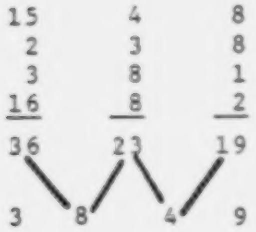
Karen, who had apparently been kibbitsing: It makes perfect sense.

Suzanne: Doesn't it really?

Karen: Mm - hmm.

Jessica: I didn't understand it.

They were then interrupted by the general discussion about what Helen had just put on the blackboard:



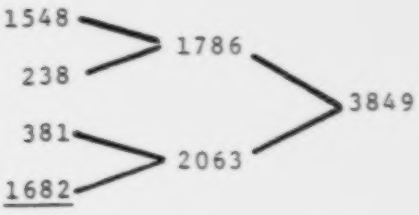
There were yet more exclamations of enthusiasm. "My God! Look at that! Love it!" I asked, "How did you know what things to draw those lines from?" "I add like that a lot if I have a big column...if I do it in my head. Cause it's easier to remember 36,23,19, and then add the middles." Heidi: "You hold those six numbers in your head and then squish 'em together in your head?" "Yeah. It's easier for me to add that column up, a single column, and then know that that's 19, and I store the 19." Suzanne: "Oh yeah, that makes perfect sense." "And then I store 23 and then I store 36. But I go the other way: 36,23,19. I know my first number and my last number are going to be the same, and the middle ones are squooshed together. ...I'd get confused when I had to carry on the top and stuff, and so I did that, and didn't have to carry on the top. I carried on the bottom, in my answer."

"What if you add another column?" asked Heidi. "I do the same thing. I might have to write down the 19 and the 23 and the - the answers, but I'll add them in my head." Suzanne: "That's clever. Really clever. You know how to add." And in a whisper to Jessica she said, "She knows what she's doing...Creative adding."

Jessica, appreciatively: "Creative adding."

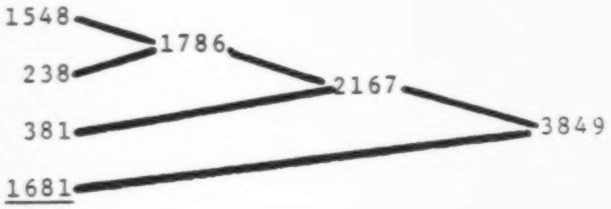
Suzanne: "Creative division, creative adding."

Maggie Lampert, the observer, was next: "Well, I don't know if this is really too different from what he did, I mean, if it qualifies as different."



"What I did was to add these two numbers together, using the system that he used on the problem in the first place." The top part had already been done, as we introduced the example. Lampert explained how she did the bottom: "Well that's nineteen hundred eighty-one, umm, two thousand sixty-one, two thousand sixty-three, - using the same system that he used up here." (Suzanne, in a whisper to Jessica: "That was the original question.") ...And then I did the same thing with these numbers [the two intermediate sums]. That's thirty-seven hundred eighty-six, thirty-eight hundred forty-six, thirty-eight hundred forty-nine."

She asked Vicky, "Is that how he did it when he got a whole bunch of numbers?" Vicky: "That's the right idea. I think I tried to do it how he did it." She was the next to write hers. "It was hard for me to do it his way. It was hard for him, too, he got the wrong answer. But - I said seventeen hundred thirty-eight, seventeen hundred seventy-eight, because of this forty, seventeen hundred eighty-six. I wrote it down, but he didn't." Then she added the three hundred eighty-one. "Seventeen hundred eighty-six, two



thousand eighty-six - and then he probably added the eighty next, but I had a really hard time with that, so I went two thousand eighty-six, two thousand eighty-seven, and then I tried to add on the eighty. That was really hard for me, but I ended up with twenty-one hundred sixty-seven, up to that point. And then I went, twenty-one hundred sixty-seven, thirty-one hundred sixty-seven, thirty-seven hundred sixty-seven, and again I went thirty-seven hundred sixty-nine and struggled, and somehow added on the eighty. Oh...I had to write that part down. And then I got the answer. I think he was doing each one, kind of adding a running total... He got the wrong answer. But he was close."

Lampert then pointed out that "In the method that Sara used, there's no carrying. Once you've got that down - I don't know if that works with other numbers. It might just be particular to these numbers. You've sort of done all the carrying basically in the steps. And so, - this is sort of coming back to the question of whether that method is quote longer. The final step where you write them down doesn't involve any carrying. So it might be quicker. But that might be peculiar to these numbers. I haven't tried another example. That is, you can do it - once you've got 2000, 1600, 230 and 19, you can do it left to right. Because you don't have to carry."

"Well your carrying is done in adding the column," said Heidi.

**Karen:** "The only time you'ld have to carry is if it came out to - 10 times - or, you know, squared of the-"

**Maggie:** "Could it?" There was a brief discussion of this question, with differences of opinion, but we did not try it out on other numbers. Later, Karen inserted that she had found that it could, with certain numbers, turn out that you would have to carry.

By then, astonished at the variety of ways, I thought we had all of theirs, and I showed my own.

$$\begin{array}{r}
2000 \\
1000 \\
600 \\
230 \\
\underline{\phantom{0}19}
\end{array}$$

The 2000 was the same as Sara's - the 2 thousands; the 1000 came from adding up 500, 200, and 300; then I stopped and added the 600 separately. The rest of it was like Sara and Marya.

Ruth then pointed that she had yet another way, and she put up these figures:

$$\begin{array}{cc|cc|l}
\cline{3-4}
1 & 5 & 4 & 8 & 50 \\
 & 2 & 3 & 8 & +40 \\
\cline{3-4}
 & 3 & 8 & 1 & 160+3-4 \\
1 & 6 & 8 & 2 & \\
\cline{1-4}
\multicolumn{1}{c}{3} & \multicolumn{1}{c}{6} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & 159 \\
\multicolumn{1}{c}{} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{9} & \underline{+90} \\
\cline{1-4}
\multicolumn{1}{c}{3} & \multicolumn{1}{c}{8} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{9} & 249
\end{array}$$

Here is how she explained it: "Well basically I added the tens and the ones. So instead of going through 48 plus 38 I just rounded it off, and just got 50 and 40. And then I added the 8 and the 8 [meaning the 80 and the 80] and then I moved around to the ones. That's 163. And I just subtracted that 4 from the rounding off. That's 159. Then I'm adding the 50 plus 40... I'm just taking care of my tens and ones. And then I moved over to the left where it says 36 and added up those thousands and hundreds. And then added the 249.

These seven different ways, and the interest they generated, were a wonderful opening for the question which was really on my mind: "What is it about the way that is supposed to be taught that makes it supposed to be taught? ...the way the math books say... I'm wondering, if a kid's doing it this way, what's special about the other way? If anything."

There was a long silence, before Vicky said, - only half in jest, "Some day if they only had a piece of paper that was with exactly enough room to put one-" Many people laughed. But somehow there seemed to be a recognition that this might be the best answer they would come up with. Suzanne protested that even that wasn't a very good answer, "Well no, really, because a lot of--. I think--.

They could do it in their heads. Some of the ways that were done were done all in the head; the means of just writing it down was just to show you what was done in the head." Helen confirmed this as far as she was concerned.

Vicky then took her thought to its extreme point. "I think there is no reason." After a brief discussion of the possibilities of teaching Helen's system by rote, Heidi said, "I think I may have an answer to why it's done that way. It's 'cause everybody does it, i.e. enough people do it so that it's a common-- piece of the common body of knowledge. ...I like the one that Katharine did...I can't think of a reason that would..." "It's applicable everywhere," said Karen. "That's what I meant," said Heidi. "Well so are the others," said two or three people. "Well, they're all applicable, but- " Heidi said, "It may be that one caught on enough, so that people did it, just for ease."

Suzanne: "One's neat. The conventional way's neat. So that you put down as few numbers as possible. And come up with the right answer." Several people found this a helpful comment, but the discussion got turned into another direction, before there was a chance to discuss whether putting down "as few numbers as possible" would make it an unlikely candidate for teaching people - for helping people understand what was going on.

Karen then showed a way she was taught, which prompted

Jessica to ask, "Do you think that kids do it this many different ways?...at whatever level one gives these kind of numbers to children, then do you think that they're really doing it all these different ways? Or do you think that they all do it the way-" Karen: "I think they're doing it all the different ways." Jessica: "You do? Because it sounds like that in our own lives. You do it your way, you do it that way, I do it the hundreds, tens, ones way-"

Heidi thought that within one school system they would do it the same way. Some disagreed with this, and Helen pointed out that she invented her way "for survival" while she was in parochial school - it certainly was not what she was taught.

Vicky: "I bet that if you look at the kids who are having trouble, they're the kids who don't have their own way to do it, and are trying to do it...the regular way."

This remark led to considerable discussion, and eventually to acceptance by a number of people. Anna, a learning disabilities tutor, said, "I think Vicky is right. The children I see that are troubled with math are really trying very hard to do it the conventional way or whatever it is that they were taught to do in the classroom, and everybody else is doing,...and haven't come up with their own method...It's almost like they don't have their own method in order to compare it to the conventional method, so that they can see, like we do..."

Suzanne added another element. She insisted on the importance of teachers' allowing kids to pursue these methods of their own. "Don't you think there are kids who sit at their seats and struggle with the problem and can figure it out but are afraid to put the numbers down because they aren't going to get the answer the way they're supposed to get the answer? Do you know what I mean? So like I would never know...unless I asked."

After considerable agreement was expressed, Vicky went on to clarify: "That doesn't mean that anyone who does not invent their own system has trouble with math... I mean,...I think that I used all the systems they told me, and that I did not have trouble with math even though I didn't invent my own systems. But I think that people who do have trouble usually have not invented their own systems."

Anna added: "Then there's the group who has maybe trouble but invents their own system - somehow is able to deal with that, comes up with some new way, which is their own system...Therefore they don't have trouble."

After a little more discussion, I asked, "I'm wondering...whether one might conclude from that that it's a good idea...to encourage kids to invent their own systems..." Suzanne, with great thoughtfulness, replied: "I would say, yes. This is a complete turn around for me I think, because last year I wouldn't have felt this way at all. I'd really stick to the conventions and get really paranoid if the kids can't understand the conventions. But I don't think it's a matter of-, I don't think what's important is that they understand the conventional rule, but that they understand the concept of what it means to add things together, or to divide things. And as long as they have an understanding of what the concept is all about, the way they manipulate the numbers to get the right answer is okay. As long as it makes sense to them. And that if, if I wanted to check out a kid's system I would have to sit down with them and make sure that they understood the concept and weren't just putting 1, 1, 1, 1, 1 up there ["carrying"] without having any ideas of where those 1's came from..." Ruth: "Do you want them to know the conventional way?" Suzanne: "No, it doesn't matter to me as long as the answer is correct and they understand how they put numbers together. For instance, I was sitting here thinking of how Helen got the 36, the 23 and the 19 to get to 3,894. And she had to understand that these two, the 3 and the 1, represented tens. And that went in the tens column, and that the 2 and the 6 represent hundreds, and that went in the hundreds column. And that it wasn't just arbitrarily why you add this number and this number to get the right answer. That she had to know what columns she was adding and what those numbers stood for."

Vicky: "It seems to me that if you can develop your own system, that is a valid system, that you could then understand other people's systems. And that that is a good thing to do. Because different ones help you in different circumstances."

Helen: "And I think also, adding on to that, you could use your system in doing the same thing in different ways, like when you're using it with fractions or when you're using it at the grocery market. I mean, then it becomes--. Every time you use it doesn't become a new thing because your system is working for you and you understand it. Then it can be used in many, many ways without having to say, 'It is the same way as you did it.' I don't know how many times I have to say to my kids 'It's the same way when you are just adding two numbers that you carry.' Or 'It's the same way when you are just subtracting two numbers.' Like when they're adding fractions with big tops. All of the sudden they'll do what Katharine did but leave it like that. What that says to me is that they understand the system when they have to add two numbers. But they do not understand addition because they are leaving that 19 there. So I say to the kid, 'You do it the same way you do it when you have just two numbers.' Then suddenly it's OK, they know how to do it. But they still don't understand addition. So that, I think if you have your own system, then you would be able to use it in anything that you did and it would work for you not only in just..."

Jessica: "Can I just say that I think the issue of effic-- I think that it's a bit of a hoax that it is more efficient. Because I think-. What's more efficient is if you have to understand someone else's addition problems who's using that system. It's slightly more efficient if you have been introduced to that thing with the marks up there on the top because then I don't have to think about it for 30 seconds, so I don't have to think about the chicken marks...like I had to think about that, what's that mean, and what's that... why's that work, you know?...That new one. Whereas I don't have to think about that other one. But I don't...I th...I really do believe that it's a ho...I know it's a hoax 'cause the only time I figured out my own system was when--the first time in my life I had to deal with the fucking attendance books and I never had to add so many numbers together in my life. And i figured out an efficient system and it doesn't have a damn thing to do with those chicken marks. You know, it has to do with thousands, hundreds, tens and ones. So I think we believe that it's more efficient. And I think that speed is the raison d'etre for our teaching it all the time. But I don't think it is. I mean, I really don't think it is."

Karen presented another point of view. "I think the kids who are having real problems, it's important to give them a way of getting into it, of doing it...There are some kids, I can sort of...throw an idea out...and they get it. They understand how to work with numbers...There are other kids who don't have a way of doing it. So giving them a way, with the tens and ones...in the same columns, gives them a way to process these numbers."

This point of view, coming late in a long discussion, was not much explored. We stopped for the break, and when we resumed, there were two other points I wanted to draw attention to in our last hour together. One of them is worth mentioning here.

I brought up another classroom example--one that Karen had presented in her journal. In her first grade class, a child had done the following problem:

$$\begin{array}{r} 42 \\ -\underline{27} \end{array}$$

in an exceedingly interesting way. When she asked how he had done it, he said, "40 take away 20 is 20; then take away a 7 and add a 2--15!" The members of the group were, once again, impressed with the degree of understanding this represented, and with Karen for finding it out. They also realized we could spend another session on possible ways of doing subtraction, but I made it clear that I had a specific point to make at this time.

In fact, when Karen first saw this example, the child's answer had been wrong. He had put 13, because he had forgotten to add the 2. When Karen asked him how he had gotten that answer, then in the course of explaining his way to her, he realized that he had forgotten to add the 2, and he corrected himself. I undertook what I referred to as "a speech" in order to develop the point I wanted to make in this last hour: "He happened to make a mistake and so she found out not only the mistake, which was rather trivial, but this fantastic way he was doing subtraction...In some discussion we had last time, when we were talking about different kinds of questions and different kinds of answers that kids give, I said at one point, 'Well, on almost any kind of answer, you can ask, 'Oh, how did you get that?' And the discussion that followed that remark of mine all dealt with cases where the kid had made mistakes. It was as if everybody was assuming that that's what you do when a kid makes a mistake, and the point is kind of a nurturing point of, 'Don't tell them they're wrong, ask them how they got it.' Whereas my point was a very different one of, essentially the only thing that matters is how they got it...Whatever the answer happens to be, what matters is, Well, what went into it? What were they doing when they got that? ...There are two reasons...why I think it's important to ask, 'How did you get that?' no matter what the answer is. One is...if you only ask how you got that when it's wrong, well, they're going to catch on pretty quick...But the other thing is that...even if they haven't done any fancy way to get the answer, if they have to sit back and think and present to you their reasons for having done what they did, whatever those reasons were, if they're sitting and thinking about them in order to be able to present to you what it was that they did, that's where they're doing the useful intellectual work."

Too many other general issues crowded our agenda for us to spend any time discussing this view at the time. Several people, however, referred to it in their final interviews as an important and influential thought.

Rather than report here on the discussion with which this session closed, I will leave the teachers' thoughts about general issues for the next chapter.

One final word about this session on "creative addition," however. After the break, at the end of the announcements, I distributed copies of Vicky's drawing of hands, "as a memento." Someone asked Vicky for her autograph. Someone else suggested, "It could be the cover of our book." "Our book on 'Math as Poetry,'" said Jessica.

## 8. POST SCRIPT

### From the Final Interviews

These excerpts represent comments from all eight of the teachers for whom this was the only year in the project. They are the ones for whom the elementary arithmetic work was a major part of the experience in the project: to a large extent, their remarks refer to this work. The remarks are arranged here in four categories.

On my purposes:

We tried really hard to look at...what we really know--how do we know what we really know...even more than how we learn... So I think that's what we did. We all agreed to sit down and talk about these different things...If we all just sat down and said, "Well, we're going to talk about how we know what we know," you can get very carried away with a lot of language... And you almost need something incredibly basic--simple--to focus on, to realize just how complicated everything really is.

I think you like to push people to think, not to just give a definite answer, but to explore why they say what they say, or explore it and then go on. I think you have a great way of asking questions that lead people to make more discoveries. And I think you like to see how people can, opinions or whatever, can evolve and come to something else. And I think you do that with us and hopefully we will do that with our kids... I think teachers are so programmed to finish what you're supposed to be doing. "Get 'em to get it right. So what if they haven't understood how they do it, as long as it's right, and they can pass the achievement test."

I don't know if it was just really imagination...it was more of an exploration. You allow people to explore the way they think about certain things, see if they can come up with different alternatives or different solutions, without ever saying they're right or wrong...You always make them feel like they can pursue it, because of what you say...I think when I went to school, the teacher would either say it was yes or no, right or wrong. If you were wrong you were devastated, and if you were right, you figured, "Well, I said it okay." You may have memorized it. You didn't care what the thought process behind it was, as long as you could spout off the right answer.

I think a lot of what you do is to build up a common ground to talk about, and I didn't realize that at first.

On the importance of time:

Something's changing for me, that I'm getting the sense of just time...I felt like I've been a little bit stuck in not being, in not really having a sense of passage of time, or learning things that require the passage of time. Like becoming an adult, for example. When I was 23 I thought I was an adult...and I'm now sort of realizing really how things change over many, many, many years. And that with teaching, you need to watch kids change over many years in order to understand what's happening to them.

Certainly in the seminar people asked...of each other, "What do you think about this?" in a way that was very broad. So I mean, I think that lends itself to never finishing anything. But it was clearly a seminar of open-endedness--just lots of room for growing in it...The more questions I learn to ask, the more unanswered questions there are...The seminar has been frustrating but stimulating. It's been frustrating because there's only a set amount of time in a set number of months, and we've just started so many things, and--we wouldn't finish them next year.

I remember one thing that struck me. Helen, she was a second year person, right? I remember one speech that she gave really kind of revealed her, the process that she had gone through over the two years, and how it really made a difference for her. I don't remember exactly what she was saying, but I, it was really great to listen to that. I think the issues, the kinds of issues that came up, or that you were trying to get at, really are long-term things. It has to do with how people see the world. It's very basic...real core kinds of things...I remember feeling that, wow, you can get a view of this whole picture, this whole long period of time, how things changed.

I think that the issue of long-range, like taking on something like observing the moon over the course of two years...I think that being able to do that kind of reasoning or whatever... is really, really important. And I think a lot of people really don't do it, and I think that's one of the reasons, for example, people are advocating nuclear power. I really do. I think because there's no clear, there's no immediate cause and effect, it's gonna be 20 years away. There's no question about it, and yet people are counting on their lives as if that's just not part of their decision-making.

Teaching is so open-ended that you never get through everything. And I'm learning to deal with that better now, and not to see it as a failure that I didn't get through, but that there's always more. You could teach 800 days in a year and still there's more stuff to do, or different ways to do it.

On their own learning:

I thought that math was either right or wrong. I never really thought about it as thinking. It was thinking, but really rote. And it isn't just right or wrong, I always thought of math as being very definite, correct or incorrect.

If I could do it in my own time, I like it...And not having to say, "I finished it, or did it." I like to do it in my own time and not have to be accountable in terms of what the other people were doing.

I guess I didn't think the course was going to have any focus on the participants as learners at all. I thought that like most courses they would be outside. you and focus on kids or issues or something, not on yourself.

I find group learning interesting. I guess being aware of all the different ways that people go about learning things. Not just adding different ways, but the different ways they really see the world and think best about it.

I really never thought of myself as a learner, now that I'm older. I guess I do, I guess in this course it was nicer to be a learner because you didn't have to learn something someone told you was right. You were able to develop what you learned, whereas other courses give you information and you're supposed to absorb it.

Sometimes when you asked questions and I didn't know the answer, I felt that there was an answer, or that I felt silly that I didn't know the answer. Then I realized that...when you ask questions you're not really looking for a right or wrong answer, so that's okay.

When we started trading backwards, that was terrific, because I had to really think about what it would mean, and why was I doing it, and what did the numbers mean. And then I went through a period...where I couldn't do it...And then I had to think to myself, well, why couldn't I do it, and what was stopping me, what happens to kids, and what's mechanical and what's learning, and--what's real learning and what's just taking on information that somebody else gives you?

At first I just couldn't understand what their notation [two other participants] meant. My first reaction was probably something like, "Well they must be wrong; there must be a mistake," and then as I compared it to mine I was able to see what sense they had made out of it.

It was really very stimulating to my thinking, on my own level, and at our last meeting, I really felt that the issue of trusting what you think you know about how kids learn is an ongoing thing, and that I feel that I trust what I think I know more than I did before. And I think it's based on the kind of work that we did.

I think there's a certain energy when a whole group is suddenly taken with an idea.

I suspect you made more thinking people of all of us.

I trust my hunches more. And even when they're wrong, that's okay...The whole business of going in there and sitting and being able to say, "I don't get it" and what you've done is take the time when people don't get it to stay with it until either people get it or they feel comfortable with wherever they're at. And I think that's really important to do...I was trying to translate that over into class work.

Anyway, I have a lot more confidence and more patience as well.

On children and teaching:

I think that in math things I'm really gonna be more likely... to say to kids..."How can you do this...what are the different ways that you can do this?"...I found that very important, the last meeting that we had...I feel that that's going to be a more integral part of what I'm doing in teaching now.

Look at how many ways we did addition the other day...There may be a right answer, but the ways that you get to that can be different. I think that's what's the most important thing in teaching. It's that...you can have a right answer but that kids can come up with their own ways of getting those answers. That's important.

Maybe because I really wanted to be able to write something in my journal I listened more to what kids were saying, or wondered why they said what they said.

One thing that I know is that the curriculum, the stuff that the kids are doing has to be really more integrated and that I have to really be paying more attention to what the kids are thinking about and what interests them as a way of figuring out what they should do in the course of the day. Rather than, you know, they have a half hour's worth of this, and 45 minutes worth of that...you know, to fill out the day, you know what I mean? (Laughter)

I learned more to watch how people learn, watch how my kids learn.

Sit and watch the kids like you did this morning...it's such a high to do that. To watch what--even what fingers go down, or the little--or the patterns of marks like in multiplying, do they make patterns in rows, or...

One thing that intrigues me is questions. What questions produce thinking? What kind of questions do children ask? How do they ask it? Do they say, "I don't get it," and when you say, "Well, what don't you get?' how do they express what they don't get? How much do you have to know in order to formulate a question?

That helped me...a way to look at the children and how they were thinking about what they were doing.

I tried to figure out whether I was using their explanations to help me teach them better.

There are benefits to sitting and talking with a child and figuring out what that child is figuring out. There are also benefits to all kids working together and experimenting and saying, "Hey, look what I found."

I'd like to be able to ask questions...and get kids to really think about what they're doing and to go on.

I feel that I've made compromises in structuring my classroom from what I really think it should be towards something that I think the general community can feel comfortable about, and I really don't think I'm going to do that so much next year. And that's for a lot of reasons, but I'm sure that that trust, or something, has helped me feel that I can, that I'm ready to really try a year of...doing what I want to do.

That whole discussion the other night of asking kids how they do it when they had the answer right really hit me, because I don't think I've done that enough...I think that I have fallen into the trap of, once they've got it, letting it go.

What I spent more time on doing was, "How did you get it?" "How did you come to that?" I think I did some of that before but I certainly do a lot more of it now, and it's something I want to do...'cause I think, what it does is, it focuses on the process of doing it and of thinking.

I was thinking about it the other day, when a woman was talking about some kids and the way they were doing their math, how terrible it was, it was all wrong, everything was wrong. And I thought to myself, I would have liked to have seen the math papers, see what they had done with it and how wrong they really were and if they really didn't understand it. And I realized that that's what I learned.

## V A VIEW FROM THE OUTSIDE

This section was written by JoAnne Gray, a graduate student at MIT, who has had experience in community organizing and in community schools. We initially asked Gray to look at some taped sessions of the seminars from the view of interpersonal relations as these seemed to be influencing the course of events. We told Gray almost nothing at first about the project or the people in it on the view that we wanted her to come to the data without any of our biases. After her first few analyses of the tapes she saw, we began working with her, gradually giving her more information, asking her more specifically directed questions concerning what she was observing, and evolving other directions that her analyses might take.

Gray was able, as we had hoped, to see aspects of our work that we had quite overlooked as a result of our close involvement with it. Her comments on the respective "styles" of each of the facilitators (Duckworth, Bamberger, Lampert) appropriately capture, we believe, some of the significant differences among us. In turn, her rather detailed descriptions of one session, bring to the surface the importance of the interactions among the teachers themselves, as a significant source of change in their thinking. And this holds for changes in the thinking of individual teachers during the seminar, as well as some of the changes that occurred later in their classrooms. Gray's work represents an additional kind of analysis of the data and also provides another dimension to the many-layered phenomena that the project generated.

* * * * *

## Introduction

The purpose of my involvement in the teacher development project was to provide an alternative perspective on what the facilitators did that made a difference in the participants' perceptions, ideas and approaches to learning and teaching. I analyzed several two-hour videotaped sessions in order to identify some of the interpersonal factors that seemed to have influenced the teachers. I focused on the intervention styles of each of the three facilitators and on how these styles affected the participants. Much of this examination centered on the contexts within which the facilitators' questions were generated, how questions were addressed, and the degree to which these inquiries either facilitated or limited the processes of teacher's confronting their own cognitive assumptions and behavior.

Throughout my involvement, I attempted to respond to the following issues:

- what kinds of questions were being asked by the staff and what was their impact on the group?
- what characterized the interactions among the facilitators, among the facilitators and the teachers, among the teachers, and what were some of the observable results?
- what were the changes in behavior in and among group members and what seemed to prompt these changes?

## Methodological Issues

I discovered, after some initial attempts to look at and to make sense of the videotapes, that it was quite difficult to absorb and respond to the enormous amount of information contained in a single two-hour session. The initial methodological consideration was, then, to develop an approach that would enable the analyses of rich, detailed qualitative behavioral data.

The procedures of "chunking" the videotaped protocols-- i.e., establishing boundaries for grouping information into patterns for analysis -- was the primary methodological technique adopted (see also Section II). After initially viewing a two-hour session in its entirety, and noting those interactions which alerted my interest or attention, I would segment a typewritten transcript of the videotape into specific noticeable patterns and themes that seemed to be emerging. The patterns and themes of interaction were illustrated by directly observable data. Several issues developed during the viewing of the taped session:

- As I looked at the same materials more than once, my perceptions of what was happening shifted slightly. For example, in one instance, I failed to notice altogether that Duckworth had made an important intervention: by interrupting one teacher's story of her experiment, she created for several others the occasion to think about and construct many of the relationships for themselves. This, in turn, alerted me to a whole new aspect of the facilitator's role that at first I simply didn't see at all. The issue of how to deal with these shifts in perception and what they represent was a constant question in my mind.
- These shifts in my perception also raised the question of how one perception interfered with or perhaps "wiped out" initial or less vivid, less worked out perceptions.

- Since the videotaped sessions provided such a rich source of data, I was aware of the fact that it was possible for me to see "new" information at each subsequent viewing. The question then becomes, when do I stop looking, when do I have "enough" information to say something valid about what was going on in the tape?

## Profiles of Intervention Styles

After viewing three sessions, each led by one of the facilitators, I was able to distinguish characteristic styles of intervention.

In the first session I viewed, Bamberger was working with a computer music system--typing instructions, demonstrating what various instructions did, and responding to the teachers' suggestions for trying out various possibilities as tests of their hunches (see also, Section III). Instructions to the computer caused a music synthesizer to immediately "play" something. In this session nearly all the instructions resulted in the "performance" (by the synthesizer) of percussion sounds. These were mostly either long or short--'S' for "short" and 'B' for "big" or longer. The task as put to the group was to think of the computer as a kind of "mind" and to try to figure out how this "mind" was "thinking"--i.e., how it "understood." The teachers were encouraged to "ask the computer to do things" in order to test their hunches and to develop evidence that might confirm or disconfirm them.

On the first viewing of Bamberger's computer session (1/16/79) I focused on the detailed explanation/demonstration that characterized her introduction to the proposed activity. Although some of the particulars in this part of the session did not, on the surface, appear to conform to a traditional "teacher-student" situation, Bamberger's approach at the beginning of the session suggested the authoritative, directive teaching mode. Similarly, the teachers displayed the passive, receptive behavior that suggested the student role. During these times, the teachers basically listened to Bamberger's introduction and watched her demonstrations, only asking questions for clarification:

"You mean we're going to use that terminal?"

"Is the 'S' on your paper there now?"

"Can you tell it how fast it should go?"

A noticeable change in this initial mode of interaction occurred when the teachers began to engage in conceptual/doing tasks that required them to demonstrate or to explain. In one case, the teachers clapped out the difference between durations of the 'S' and 'B' of the computer. Bamberger's interaction, at this point, shifted from explaining and demonstrating to a more direct technique of encouraging the teachers to experiment and to give concrete verbal accounts of their thinking. This elicited more explanatory and descriptive statements on the part of the teachers. The kinds of questions that Bamberger began to ask were:

> "How would you draw a picture that would show the difference between the S's and the B's?"
>
> "Do you want to put them on the board?"
>
> "Anybody have anything different?"
>
> "Who's got an idea?"

I would characterize these questions as "What do you think... tell me" kinds of questions that are designed to elicit experimenting and describing the results. The teachers' responses reflect this mode of questioning. Bamberger also actively tried to translate or restate more clearly some of the teachers' ideas and thoughts. This process of discovery and "giving reason" represents a style of intervention that was evident throughout the remainder of the session. I described this process as follows:

> Bamberger poses a "what if..." or "what about" context for inquiry.
>
> The teachers respond with "I think..." or "It seems like..." statements and questions.
>
> Bamberger says or does something that attempts to illustrate the teachers' statements or questions. She does a kind of let's see...try this...now what do you think?

> The teachers respond with "I think..." or "It sounds like..."

The teachers also began to actively suggest things for Bamberger to do to experiment with their ideas to make them more concrete and visible. During these exchanges, the teachers asked questions like:

> "What if you [Bamberger] typed it without an 'S', without a space, at all?"
>
> "What does it sound like if you don't put space bars in?"
>
> "Can you tell it to play it back to you without space in between?"
>
> "You could play it the way you played the first one?"

Bamberger picked up on these suggestions with statements like:

"Tell me what you would like to do..." providing more possibilities for the teachers to confront their own experiences and thinking.

Another significant shift in activity occurred when Bamberger engaged individual teachers to explore their thinking about and understanding of the task. Bamberger addressed individual teachers by name, and designed her interaction to push the teachers to reveal more of their thoughts and ideas:

"Now the questions what -- well Isabel's question?"

"Do you remember when this came up before? Actually, it was you Jessica, who decided it was so neat."

"...and it does basically what you said, Helen?"

As this direct interaction occurred, individual teachers more readily offered their own thoughts and suggestions which in turn prompted other teachers to react to them directly. This direct involvement also seemed to encourage teachers to discuss the tasks with other teachers. For example, when there was a conversation between specific teachers and Bamberger, the level of interaction among the teachers also increased. They would talk directly to one another rather than directly to Bamberger, asking questions among themselves like "What do you think...?" and "Did you mean...?" This behavior suggests their willingness to own the task and to take risks in knowing and understanding it. One particular example of "owning" the task and taking personal risks in exploring the topic, occurred when the teachers realized that they could, in fact, "teach" the computer, themselves. Instead of merely accepting Bamberger's statements that this "teaching" was possible, there was a real change when the teachers finally actually began to do this teaching on their own. The point at which this became most evident was when Bamberger, on a teacher's suggestion, made a new program that she called "Jessica." During this part of the session, the teachers seemed able to generate much more sophisticated ways of talking and dealing with the computer. Comments such as

> "You've been reinvented..." (i.e., "Jessica")
>
> "It's a funny way to use language..."
>
> "When you talk about computer language...it's not language, almost,..."

suggest that they have gone beyond the sense that the computer is something suprahuman, but rather a machine that can be manipulated and controlled by people.

Bamberger always seemed to be on top of the session and its agenda. At first Bamberber's interventions consisted primarily in lengthy explanations and descriptions of the task. The teachers' active involvement during this period was limited to listening, with minimal participation from them and few exchanges among themselves and with the facilitators. At a later point, Bamberger shifted her interventions from the initial lecture format to a mode that encouraged the teachers' descriptive, reflective responses about the computer and how it was "thinking." Bamberger's strategies in this mode were questions that encouraged individual and group accounts of reasoning and thinking. The teachers became more involved in active, mutual demonstration, questioning and discussion.

Several of Bamberger's interventions also involved "bootstrapping" when activity seemed to be stuck or not moving productively. At these times she was active in re-framing the issues in response to what was happening and why. For example, at one moment, a point of confusion arose that seemed to be holding up the discussion. The session was "stuck." Bamberger intervened by saying "I guess I'll have to tell you more..." This comment suggests that while she was willing to allow the teachers to struggle with ideas and concepts, she had clear limits in mind about how far to go with this process in terms of eventually gaining clarity. The example also suggests that Bamberger maintained implicit notions of the direction and purpose of the session that informed her sense of how and when to intervene and whether and when the session strayed too far from a productive path.

## Duckworth's Session (3/20/79)

Duckworth's session was a continuation of the teachers ongoing observations of the moon. It was the fourth session in which these observations were being discussed. While the teachers had been given some minimal suggestions about how to make observations of the moon's movements (look at the same time each night, look from the same place, etc.) no other information concerning the solar system had been given to the group.

My overall impression of the exchange during the initial sequence of Duckworth's session on the movement of the moon, concerns the absence of the traditional "teacher/student" mode of interaction that appeared to dominate the beginning of Bamberger's computer session. Duckworth and the teachers seemed to be mutually engaged in an exploration in which Duckworth's leadership was more suggestive of participation than instruction. Duckworth was involved with the others in reporting her observations, not as definitive answers, but as her own individual perceptions and observations. She presented these perceptions as no more "correct" than those of the others. For example, here are some of Duckworth's comments during this sequence:

> "Saturday night I drew it looking like...is that possible?"
>
> "That's funny, because it looked like that to me..."
>
> "I didn't think of that..."
>
> "I think, but I'm not certain..."

The collaboration between Duckworth and the teachers seemed to facilitate the teachers' ability to respond. Most of the discussion during this sequence was rich in the exchange of information and free-flowing. With one exception, all the teachers were actively involved. The session also seemed to have a life of its own, as though its momentum was self-generated and self-sustained. For example, the discussion initially began with the teachers' personal observations of the moon (color, position in the sky, changes from time of day and position, etc.), then switched to one teacher's comments on a child's perception of the moon as being the planet Jupiter, and finally ended in an exchange about how to help third and fourth grade children understand the concepts of size, space and distance. The transitions from topic to topic were not labored, and the conversation did not "suffer" from the shifting flow of topics. Duckworth's intervention style seemed to encourage this mode of interaction.

I was also intrigued by the types of questions that Duckworth posed during this sequence and the responses that these questions elicited. Duckworth's questions were generally the kind that I would have expected to elicit very short answers of the "yes/no" variety. Some examples of these kinds of questions are:

"Can you draw its shape?"

"Has anybody seen the moon orange that high?"

"Is there anything that anyone would think to ask?"

"Any other kind of question that you would ask?"

Instead, Duckworth's questions prompted responses from the teachers that were highly detailed, personal and rich with subjective data. Their answers contained a high degree of experiential information about the events as opposed to an objective "reporting" of the facts of the events.

The group was engaged in an exploratory, experiential activity that allowed them to see the old and familiar in some new and exciting ways. The experience of making and shaping old ideas into new was grounded in their own experiences based on their abilities for observation and understanding.

Duckworth's style of revealing her own observations for analysis and disconfirmation established a context that was conducive to a high level of insight-making from the group. She didn't present herself as an expert in the experience which facilitated the willingness of others to explore and to risk.

Duckworth was very explicit and active in engaging people in non-threatening but direct ways. From the beginning, Duckworth addressed individuals by name, pushing people to respond directly to her inquiries. Although she was quite direct in challenging teachers, she also seemed to be sensitive to those times when it was most useful to confront their perceptions and feelings about the topics at hand. For example, her inquiry, "You like that idea, Diane?" resulted in a lengthy and interesting exchange between Duckworth and Diane. I'm not certain how Duckworth knew that Diane was ripe for that particular kind of engagement at that particular time, but she was quite adept at bringing Diane and others into the action in several instances. She seemed very much aware of people and "where they were at" during this sequence, and her pushing and prodding worked just right.

Duckworth's intervention style was also noticeably less obtrusive than Bamberger's. Her introductions to activities and tasks tended to be more concise and brief. She limited the information that she gave about the "whys" and "hows" of the tasks. The teachers were more on their own to figure out the significance of the information they got from observing the moon.

## Lampert's Session (4/3/79)

The session that Lampert led centered around a story of a social studies lesson from one of the teacher's classrooms. The lesson had been concerned with the Roman Empire period and, in particular, Jesus as a figure in it. This seminar session could be characterized as quiet, methodical, disciplined and controlled.

In contrast to the other two, it was largely conversational and reflective. While the other two sessions did not exclude these modes of interaction, they were distinguished by their greater emphasis on physical activity (demonstrations, experiments, experiential learning) such as the computer experiments and the re-creation of the position, movement and appearance of the moon.

Lampert presented herself in a relatively cool manner that differed from Bamberger's and Duckworth's approaches in their sessions. During most of this session, Lampert was clearly the leader.

One technique of her intervention style was probing topics and ideas with individual teachers. During these times, Lampert engaged an individual in a one-on-one dialogue that seemed to focus the entire group's interaction on the substance of the conversation. For example, Lampert was persistent in pushing Lee to talk about the "Jesus example" from her sixth grade social studies classroom, and for long stretches of time, the discussion centered on Lampert and Lee. Lampert engaged Suzanne and Jessica in similar exchanges. Her pattern during these exchanges was to establish the sequence of events in a particular situation, and then to interpret the import of the situation on the teacher and the student.

Lampert's interventions also seemed to be of two general types. The first can be described as reporting questions which elicit descriptions of particular situations in detail. Examples of these interventions are:

> "Are there any more examples of situations where you had to give reason to something a child was doing in class?

"Any specific social studies examples?"

"Do you want to talk some more about that?"

"You said some words about readiness."

Once Lampert established a sequence of events in a situation from a teacher's classroom, she introduced interventions that elicited the teachers' ideas and conceptualizations about it. These interventions were framed in the following manner:

"How can you figure out what a child understands before he is asked to try to understand more?"

"Does it matter?"

"What if he had said nothing?"

Lampert identified an issue or problem, and then prompted teachers to elaborate by framing her inquiries around the teachers' thinking and understanding.

Looking back at the style of the three facilitators it is important to note that differences in individual intervention styles were influenced in part by the goals of the different sessions. Bamberger's and Duckworth's sessions seemed "warmer" because specific tasks were the foundation of each of these sessions. Lampert's session seemed "cooler," because it was designed primarily as one of relating stories from teachers' classrooms -- i.e., it was a discussion without an activity. Nevertheless, it seems clear that distinct intervention styles are reflected in each of the sessions. All three facilitators included in their repertoire the technique of "giving reason" as a major part of their individual intervention styles. Each facilitator spent a significant part of her session inquiring along with teachers into thinking and understanding. Bamberger devoted chunks of time at center stage explaining, while Lampert probed for details, and Duckworth allowed the teachers to ponder, search and wonder, but each of them shifted into a framework of interaction and inquiry that encouraged the teachers to think about and understand their own thinking. Although Duckworth seemed more grounded in this mode than either Bamberger or Lampert, all three emphasized it in their sessions.

## Interplay of Three Facilitators

A second area of analysis concerns the interaction among the facilitators within the sessions. During most of the computer session, Duckworth supplemented Bamberger's explanations and descriptions with questions and comments that served to extend Bamberger's discussions.

During these periods Duckworth asked questions and made comments that prompted further thinking and discussion by teachers:

> "What made you think that?"
>
> "Faster, compared to what?"
>
> "What do you mean?"

For much of this particular session, Bamberger and Duckworth displayed an ability to anticipate those times when further explanation was required or when the teachers seemed "ready" to talk more about their thoughts and ideas. In this way, both were able to blend their ways of facilitation and to draw individuals into the discussion. In this computer session, one of the teachers commented on Lampert's role as observer:

> "Lampert has the unique challenge of seeing how many times they do that to how many times we do ours."

Even when Lampert made a shift in her interaction with the group, she was seldom directly engaged by either the other facilitators or the teachers, when she posed her questions or comments. It was only toward the end of the session when Lampert, who first emerged as an observer, became more of a participant, that there was any extensive direct conversation with her.

Bamberger's interaction during Duckworth's moon session also was quite interesting. Unlike her behavior in the computer session, in which she seemed to have a clear idea of the direction and conduct of the session, Bamberger was very much a participant in this session. She very openly described her groping and searching for answers, but she did so in ways that pushed the group forward. For example, at one particular point, the discussion seemed to get stuck when Duckworth asked if anyone in the group wanted to move any of the pieces in the mock-up of sun, moon and earth, to show the sunrise. When the response was slow and Duckworth was groping to restate the question, Bamberger plunged into her own spontaneous description and hands-on demonstration of how she was conceptualizing the sunrise. Although Bamberger's explanation and demonstration by no means suggested expertise in the matter, her manner of revealing her ideas and her genuine confusion seemed to anchor the discussion in something concrete that could be reacted to and talked about more easily by the group. Bamberger's expertise comes not from knowing the answers, but in making her own thoughts concrete and "graspable." During her explanation, Bamberger manipulated the plastic ball (moon) and used this in her explanation which was reminiscent of her behavior in the computer session. She provided both an interesting set of ideas for consideration and an example of how someone can confront head-on very complex ideas and concepts.

During the demonstration of the sunrise and the sun's rays, Duckworth continued to query people directly about their perceptions and feelings and Bamberger continued to facilitate the group by anchoring the discussion through her way of telling her own thoughts, confusions and personal experiences around the issues being discussed.

Bamberger's comments seemed to serve as a point of departure for the teachers that enabled them to either agree with, contradict, or to react more easily and simply because of its anchoring qualities. Her statements seemed to provide on one level the basis for a discussion which encouraged the teachers to consider their own thinking on specific topics, especially when Bamberger's ideas challenged or contradicted what another individual might think or believe, and on another less explicit level, an example of a different way of learning.

Bamberger and Duckworth, in varying degrees, demonstrated their willingness and capacity to change roles. At one time, they may be presenting a topic for consideration; at another time, they may be helping the group or an individual to probe and reflect on their own understanding; and at still another, they may "be" one of the teacher participants -- experimenting, thinking out-loud along with the teachers, in ways that suggested possibilities for offering one's ideas for exploration. The experience derived from Bamberger's and Duckworth's shifting roles and willingness to puzzle over problems may have provided the teachers with a new way of understanding the facilitators as teachers and, in turn, themselves as teachers within their classrooms.

Neither Bamberger nor Duckworth intervened until well into Lampert's session. Lampert spent the first part finding out what material there was in the teachers' examples and how it could be explored. Once the "Jesus example" became the focus of discussion, Duckworth ventured a question to Lee, "Would you go about it differently?" and Lampert incorporated the intervention by asking, "What about Duckworth's question?" After this point, both Duckworth and Bamberger became more actively co-facilitators of the session, and the focus of the discussion shifted between Lampert, the teachers, and the two co-facilitators.

During this part of the session, Lampert's role became less one of a leader and more of a co-facilitator. There were long periods of time when the teachers were engaged in conversation among themselves, and not, as had been the case up until that point, largely with Lampert. Duckworth's interventions during the rest of the session consisted of questions like "What do you think that comes from?"; Bamberger asked questions like, "What would happen if you had students compare the two answers?" While the facilitators did not actively solicit assistance from each other during the sessions, they did offer support by extending and reflecting the session leader, each in her own way.

## Extended Example

The various modes of facilitation can be better understood by presenting, in contrast, an example where the participation of the teachers was more fully indicated. This example reveals not only the responses evoked by the interventions of the leaders, but also the ways in which the participants interacted with each other. Of greatest interest is the way the teachers, themselves, practiced in their interaction with one another the same styles of intervention as that of the facilitators. For instance, in the first example presented here, it was Suzanne's and Jessica's interventions that were critical to Lee's reconstruction of her understandings. This example was drawn from Lampert's session.

The incident began with Lee talking about her unit on the Roman Empire. She was puzzled by a student who asked if the Jesus as historical figure was the same Jesus as the Biblical figure. Lampert asked:

> "How could you as a teacher figure out the answer to your own question: what does a child understand before we ask him to understand more? Is that your question?"

Lee suggested that this was also part of a larger question. Implicit in Lee's thinking was the idea that a teacher should have some way of knowing in advance what a child understands and does not understand. If not, by the time a child asks a question that suggests misunderstanding, something in the teaching/learning process has already gone awry. Lee's comments suggested that she did not feel good about children demonstrating that they have not understood: teachers should have some way of anticipating such misunderstandings ahead of time.

The process that then took place involved reflecting on Lee's understanding of this example from her classroom. Jessica emerged as a significant catalyst in the initial phases of this process. At Jessica's encouragement, Lee elaborated on her understanding of the situation.
The mutual exploration of this understanding was initiated when Jessica asked Lee, "What did that mean to you - that

he didn't get what you had been talking about?" Jessica's critical question elicited a rather lively exchange (several people seemed to be talking at one time). Suzanne suggested that the boy's question ("Is that the same Jesus as in church?") might be a good one. She recounted an example of a similar incident in her class during a discussion of the Civil War. Jessica suggested that instances where children express misunderstanding are quite common and challenged Lee's negative attitude towards the situation. She offered a different construction of the situation: The experience could be seen as one in which children's questions help them to make connections and to learn.

Lee began to reconsider how she had experienced the situation:

> "...maybe I should feel good about the situation...?"

Suzanne asked a further critical question:

> "Does it matter?"

This presented Lee with quite another way of looking at the experience of children demonstrating misunderstanding. Lee seemed both confused and threatened. She said, hesitantly, "...well, I don't know...," and redirected the question back to Suzanne:

> "...do you think it matters for the child who doesn't know how big the sun is...?" (a reference to an example from Suzanne's class)

Suzanne responded with more detail: it doesn't matter if the child does not know the right answer as long as that child is making a "good" question. Lee interpreted Suzanne's comments by posing another question:

> "...the fact that he was so far off didn't bother you?...it indicated a starting point, not an ending point...?"

Through contrasting examples provided by the teachers and the mutual process of reflectively "conversing" with Lee's ideas, Jessica's and Suzanne's stories and questions helped Lee to probe her own understanding. Jessica's and Suzanne's supportive participation first prompted Lee's confusion, and later led her to restructure her earlier views of the classroom situation. At first, the very different ways of interpreting and dealing with children's misunderstanding seemed to pull the floor from underneath Lee's own assessment of the situation. Later, Lee was able to reconstruct her way of making sense of children's misunderstanding: She came to see a child's question not as a mistake or an ending point, but as a start for developing something new.

The discussion shifted once again with a question by

Lampert:

> "It seems to me you started out saying that the curriculum is structured in such a way that in the 6th grade we are supposed to start teaching about the Roman Empire and what are we assuming...or were already assuming...is that kids can handle the Roman Empire at that age. Do you want to talk about that with regards to curriculum?"

Lee responded by stating that "...it's not really a curriculum question, I guess...," and Lampert left that focus and tried another:

> "There were also some words in what you said about readiness. I wasn't sure in relationship to what Suzanne was saying what you think about that?"

Lee's response at this point seemed to recapitulate her earlier assessment of the classroom lesson. She repeated her original concerns almost as if the alternate constructions for making sense of children's misunderstanding were not on the table. Jessica recounted another story that she thought was like Lee's story. She told about her class's surprise at seeing cities, cars, etc. in Africa. As a result of the surprise, a student made a new connection that Jessica had been trying to bring out. Lampert asked Jessica to explain what would have happened if the student had not made the connection that Jessica had been hoping for.

This discussion seemed to confuse Lee's previous consideration of alternate constructions of children's misunderstanding. Although Lee had been able to understand other ways of thinking about children's questions, she had not yet reached the point where she was really ready to apply another way of thinking about it to her own experience. Lampert's probing questions prompted Lee's initial concerns to resurface. This phase in the discussion, then, seemed to be a holding pattern:

> Lampert was asking the same questions in a different way.
>
> Lee was still struggling with her own conflicting feelings about the experience of children misunderstanding and how to deal with it.
>
> Jessica was telling a story that she felt was the "same" as Lee's Jesus story, but this didn't seem to help.

It is important to notice, here, that the experience of reflecting on understanding was not, in this instance, a simple linear or additive experience. Lee alternately ventured forth to grab hold of a new view and, then, backsliding hung on to her old constructions. The experience was, for Lee, one of "pulling yourself up by your own bootstraps." It seemed to involve an enormous degree of personal risk that was not easily accomplished nor was it a straight-forward process.

As Lee continued to struggle with her own ambivalence, Lampert asked Suzanne if she had any examples of kids asking the kinds of questions Lee was concerned about. Suzanne responded by talking about kids whose questions seemed at first to suggest they are not "up to grade." She seemed to repeat her "Does it matter?" notion:

> "...I don't care if they remember...remembering has nothing to do with social studies..."

Lee made a foray into the new stance once more by considering Suzanne's reading of the situation:

> "You mean, the experience is not a bad one as long as children make connections...?"

Lampert responded by interpreting further what Suzanne meant:

> "...it lets you know what he's thinking..."

Lee, once more, tried to assimilate the new view -- she referred to these moments now as "check-in points." She had begun to develop her own sense of how to use children's questions in a very different way than she had initially. The discussion continued with Lee's substantive reexamination of her original experience: The child's question could be reconstrued and used as a means to go forward with him, as opposed to it being merely a sign of misunderstanding and, as such, a dead-end.

This particular session illustrated several issues:

- "The curriculum" can be seen as setting expectations for what children ought to be able to understand.
- Risks are involved in relying on and being responsive to individual responses and interactions as the focus for learning.
- Different constructions of the same experience can have an impact on how we think about ourselves and the world.
- The teachers in the group -- in this case, Suzanne and Jessica -- developed a productive questioning approach with each other and with respect to their own understanding of their work.

Lee's experience can be viewed as a prototype of what also happens with children. Lee, like the children in her social studies class, was coming to see her initially negative experience with children's misunderstanding as an opportunity to engage in a very different kind of teaching and learning process. She was making new connections between varying constructions and aspects of the same experience just as the boy who made new connections between "Jesus in church" and Jesus as a figure in the Roman Empire.

This session also illuminates the complexities involved in making these connections: Although Lee seemed immediately to understand the alternative explanations and constructions of the situation that were offered, she had to examine and reexamine her grounding before she could find a way of understanding that she could make her own (she called them "check-in points"). Lee's experience in this sequence also demonstrated the risks that teachers take when they question the dictates and guidelines of the "curriculum." The Roman Empire curriculum suggests that students in the sixth grade are ready to understand and learn the material covered in this unit of study. Initially, Lee thought that because her students did not understand the connection between Jesus in the Bible and Jesus in the social studies lesson, her students were not ready or had not "achieved at the appropriate level." Her reluctance to see other ways that children can demonstrate readiness made it difficult for her to see something positive in what was happening with her student when he asked questions that she construed as demonstrating "misunderstanding." However, Lee finally came to see that such questions could also be interpreted as readiness to engage new information. Rather than simply a disparity between the expectations of the "curriculum" and the child's readiness, these questions could be seen as linkages the student was trying to make indicating his capacity to engage the new ideas that were before him.

In a similar way Lee was to make her own linkages with the alternate constructions that the other teachers suggested. Indeed, she finally made sense of the alternatives proposed to her only when she could do so in her own terms. It was then that Lee could begin to develop these alternative posibilities in concrete and practical ways.

Lee's willingness to risk restructuring her thinking about kids' "misunderstanding" was encouraged and supported largely by interventions of other members of the group. This example demonstrates, then, how the participants were also able to take on the role of facilitators in helping one another to think about and gain insight into their own thinking. This was another kind of learning that later carried over into their classrooms, too.

## VI CLASSROOM OBSERVATIONS

Of the eight teachers we believe were significantly affected by the project, five are currently teaching full time. This section consists of a report on classroom observations and interviews with these five teachers, carried out by Lampert. Following some general comments on the observations, there is a series of charts in which various aspects of each of the teachers' classrooms is summarized. The section ends with a full report of one of the classroom visits.

* * * * *

### Introduction

These charts include descriptions of classroom practices among five of the teachers which seem to have resulted from the teachers' participation in the project. They are based on comparisons among classroom observations during the first and second years of the seminar meetings and observations during two years following the end of the seminar phase of the project. I visited all of the teachers during different times of the day and the school year, and in some cases in radically different school settings.

The topics chosen to categorize observations are based on an assessment of areas in which the project has had some impact. There are, of course, overlaps and in certain cases, these are indicated on the charts. "Language Curriculum and Instruction" is meant to include primarily reading and writing. It is considered distinct from "Mathematics Curriculum and Instruction" because there are several classroom practices which can be associated with one category or the other, not because children's thinking about language and mathematics can be readily categorized.

Comments in parentheses describe areas where the teacher did not change in an area found worthy of note. I.e., it seems as though participation in the project should have made a difference in this area, but did not.

Starred comments * indicate an impression that the teacher being described has been doing things that way all along--not necessarily affected by the project, but congruent with it. These practices raise an interesting question about "effects" of the project. It may be that the teachers did and would continue to do these things without having participated in the project. It may be, however, that they continued to do these things because the project gave them some sense of why they were important. If they had not participated in the project, they might have given up those practices which did not make sense to them otherwise, as well as those practices which required an extraordinary expenditure of energy and initiative. Thus, these are included because it may have been the project which provided either rationale or support for continuing practices which the teachers might otherwise have abandoned.

The goals of the project could be interpreted in terms of three levels of effect on the teachers' classroom practice. At the first level, 1, are those practices which indicate that the teacher has a new or renewed respect and appreciation for the child's way of understanding something in contrast to her own way or the way represented by the formal school curriculum. (We have called this "giving reason.") At the second level, 2, are practices in which the teacher not only acknowledges the child's perspective, but does something to extend or expand the child's way of thinking about the phenomenon at hand. And finally, at the third level, 3, the teacher helps the child to make a connection between the child's way of understanding and the conventions associated with the school curriculum. These levels are used in the following charts to analyze particular practices.

Our seminar discussions with the teachers seem to suggest that they found this third level most difficult to achieve in practice. For example, a teacher may find it quite difficult to help a child make connections between his/her understanding of some problem and the way that problem is described and/or solved in the formal curriculum.

Whether a teacher, in working with children, is trying to make such connections, is a difficult question to answer. Yet it is clear that in some cases there was no attempt to do so while, at the same time, the children's ways of understanding were being acknowledged by the teacher. It also seems that some of the teachers have become aware of the sense in which work in the seminar could aid them in helping children to integrate their own understandings with the curriculum. However, this awareness has raised questions in their minds about whether such connections could be made within the structure of schooling as it is now organized--e.g., with children learning in groups; where the teacher knows "more" and is, therefore, an intellectual as well as a social authority; and where success outside of school is built on success at school-defined tasks, etc.

For Helen and Diane, arriving at Level 1 meant a big change in the way they interacted with children as teachers. Jessica still seems to be struggling with trying to be on level 2, and yet many of her practices could be interpreted as at level 3.

For Jessica, Suzanne, and Vicky, level 1 was the starting point. They were there before joining the project, to different degrees. Suzanne and Vicky do a considerable amount of their teaching on level 2. Suzanne and Vicky are most actively struggling with the meaning of connections between the child's thinking and conventional school curriculum. They are most concerned with making sense of situations where the child's understanding and the curriculum seemed unrelated or incongruent.

Diane: Grade 1, School: Working Class

| Classroom Organization, Time/Space/Social Structure | Language Curriculum Instruction | Mathematics Curriculum and Instruction | Interaction with Individuals | Interaction with Groups | Thematic/Interest Related Curriculum | Self-Reflection |
|---|---|---|---|---|---|---|
| Many children's drawings on the walls, children encouraged to draw rather than to "color" already drawn pictures, also to cut and paste collages ①<br>Children's desks arranged in groups not facing teacher's desk. ①<br>More "choice time" available, although the choices are commercially-made reading and math games of limited scope and generally poor quality, no attempt to direct choices ①<br>More "talking" among children is tolerated during times when Mary works with a small group. ①<br>I am treated as a person with a purpose, whereas in early visits my integration into the classroom was more formal and impersonal--kids now encouraged to talk to me. ① | Children encouraged to use phonetic spelling to make "labels" which tell teacher about what they have created out of blocks, clay, etc. (Mary said she used to have them "discuss" these, but this became too cumbersome.) ①<br>(Still a good chunk of time spent on phonics workbooks, kids told to color the pictures when they finish work and spend a lot of time on this. In reading groups (all morning) Diane sticks to her agenda, doesn't pursue kids' thinking.) | (None observed in any class visits except children doing workbooks and using highly structured games and materials.) | More careful listening to children, with the assumption that what they have to say expresses their view of the world. ①<br>A considerable amount of "unproductive" behavior is tolerated ①<br>Kids do not seem surprised when I ask them "how did you figure that out"--they give relatively articulate explanations. ① | Less of a "teacherly" tone of voice--a sense that she is addressing other people, not categorical "first graders". ①<br>Allows whole class to be "distracted" from their work to something interesting happening out in the hall, spends time discussing it with them, (but puts emphasis on a conventional explanation rather than kids thinking--she sees herself as the "explainer", even though she also listens to children). ① | Evidence of work on dinosaurs organized around open-ended problem-solving, some child-made displays, some teacher-made and commercial with "content", i.e., names of dinosaurs, when and where they lived. (No clear connections between child-made displays and "content".) ①<br>(Overall quality of involvement with materials is low--i.e., there is a lot of throwing blocks, moving from one thing to another, smashing clay balls, etc.) | Sees herself as really different from other teachers in her school and relationships are strained, e.g.,<br>- noise<br>- phonetic spelling<br>- drawing<br>- desks not facing front<br>Sees biggest effect of project as an increase in her self-confidence. ①<br>Sees herself as trying much harder to listen to and understand what kids think, but says she needs to learn more to take the next step: to use what she understands ②①<br>Also feels her questions might be off the mark, so she will "just listen". ①<br>Sees herself now as "pulling out" not "shoving in". ① |

Helen: Grade 4, School: Alternative Public School

| Classroom Organization, Time/ Space/Social Structure | Language Curriculum and Instruction | Math Curriculum and Instruction | In with Individuals | Interaction with Groups | Thematic/ Interest Related Curriculum | Self-Reflection |
|---|---|---|---|---|---|---|
| Lots of active talking among kids about their work--high energy and investment (Helen works with one small group, while others work independently on well-organized, open-ended assigments). ①<br>Increased use of other adults to diversify curriculum and instruction. ①<br>Kids personal issues are a much more obvious and formalized part of classroom life. ①<br>Space much more diversified into different kinds of work areas. ①<br>Both project work and "skills" work are occurring both in a.m. and p.m. ①① | Much more project-related reading, writing, and speaking. ①①<br>More "discussions" among kids, where they are expected to listen to one another and build on each other's ideas. ①<br>Lots of use of dictionaries and encyclopedias. ①<br>Substantial responsiveness to how kids are thinking about the meaning and structure of the language--questioning to test the limits of their understanding. ② | Use of conventions in this area--teaching algorithms without any explanations.)<br>More use of manipulative materials for problem-solving activities. ①<br>(Use of worksheets is heaviest in this area--also, organized around finishing directed activities as a preliminary to open-ended exploration.) | Lots of attention to whether kids are involved in what they are doing, concern with what "grabs" them. ①<br>Classroom is organized so that Helen can spend big blocks of time in conversation with one student--pursues child's thinking and extends it; she helps kids to focus, but does not cut off their energy. ②<br>(Sees herself as less able to do this than she would like in current school because she needs to spend time on conflict resolution.) | Much more enthusiastic and personal--balanced between being a member of a small working group and its leader. ①③<br>Although Helen retains control over the whole class, there are serious periods spent on exploring whatever anyone in the group brings up for discussion. ① | Many more substantial projects in evidence around the room. ①<br>More use of human and material resources to develop ideas in science and social studies, reflecting the idea that the teacher doesn't have to know all about something in order to have kids learn about it. ①<br>Emphasis on learning about things close at hand, observable, and of concern to the kids. ① | Wants to do things with kids that she can learn from as well.<br>Sees self as "calmer" about her ongoing concern for kids to develop their own thinking a concern which she feels has "always been there" but is much more in action now.<br>Thought she became a "great teacher" at last school then confidence was shaken by experience at alternative school--felt need to prove herself--came out of it by carefully reflecting on each kid, lots of emphasis on their emotional needs, "why they act the way they act". ①<br>"Most important to me is to try to understand what they are asking me." ① |

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Jessica: Kindergarten, First grade, School: Magnet Program

| Classroom Organization, Time/ Space/Social Structure. | Language Curriculum and Instruction | Mathematics Curriculum and Instruction | Interaction with Individuals | Interaction with Groups | Thematic/ Interest Related Curriculum | Self-Reflection |
|---|---|---|---|---|---|---|
| *Uses a variety of rich materials whose use has thought provoking questions "built in" (e.g., marble roll, blocks, sand and water table with tools for measuring, etc.) ②<br>Adjusts assigned work time to accomodate children pursuing activities that are "on their mind" ①<br>*Classroom is organized to have a number of different things going on at one time. ①<br>*Accomodations are made in the schedule to spend extended periods of time on activities in which children are productively involved. ①-③<br>*Classroom organization allows her to spend time with one individual and she does this a lot ① | More use of children's own stories as a basis for assessing and developing reading and writing skill. ①③<br>(See Thematic Curriculum also)<br>*Discourages the use of materials which put the child in a passive position. ①② | *Sets problems for children in building, drawing, grouping, counting which can be solved in many different ways. ①-③<br>Responds to children's activities with an obvious awareness of the time, space, and number concepts they are working on as they work to solve problems they set for themselves. ③ | Observes what a child is doing and then asks a question which would lead the child to think about the phenomenon at hand in a new way. ②<br>*Asks children questions which require them to observe a particular aspect of an activity in which they are engaged and verbalize their understanding of what they see. ①②<br>Often asks children "How did you figure that out?", when they answer her questions (her construction of follow-up questions is limited). ① | *Uses morning meeting to "process" issues which concern individiuals or small groups ①<br>Builds group instruction around problems that arise in individual projects, asks each child how he or she would "solve" the problem. ① | *Builds writing, drawing, and counting assignments around children's involvement with materials they have chosen (e.g., labels, maps of structures, etc.) ①③<br>(This could be understood either as making connections between kids' activities and formal skill development or as imposing the teachers' agenda regarding things the children themselves would never choose as a way of thinking about things.) | Renewed commitment to the idea that teaching is difficult intellectual work ③<br>Increased appreciation of the difficult thinking involved in arriving at a basic understanding of something ②<br>Greater tolerance for and understanding of the variety of ways teachers can work; better sense of the common dilemmas that run through the teaching process no matter what methods are used ③ |

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Suzanne: Grade 4, School: Magnet Program

| Classroom Organization, Time/ Space/Social Structure | Language Curriculum and Instruction | Mathematics Curriculum and Instruction | Interaction with Individuals | Interaction with Groups | Thematic/ Interest Related Curriculum | Self-Reflection |
|---|---|---|---|---|---|---|
| A small increase in kids' input in choosing projects that everyone will work on together. ① | Acceptance of grammar and spelling errors in composition meant to "get a lot of good ideas down on paper". ①-③ | Much more of an effort to use manipulative materials to demonstrate and evaluate whether kids understand the concepts behind computation. ①-③ | Increased expression of trust that kids can figure things out for themselves, make sensible decisions about how to complete a project, etc., ① | Confidence in her ability to be the classroom leader, and juggle many different interests and activities, has increased--more skill at using human and material resources to further kids' interests and especially to coordinate it all into a class-size production, where each child has an appropriate and challenging piece of the action. ①-③ | *Projects seem to be a productive balance between teacherly organization and direction, and lots of child participation at all levels. ①-③ | Sees herself as more careful about thinking through what kids need to think about, how they would solve problems associated with the projects she gives them - results in more well developed directions, for example. ① |
| *Much emphasis put on kids taking responsibility for their own behavior problems--talking things through, etc. ① | "meaning" of what kids read is stressed more. ① | More attention to the process of working a problem out. ① | *Extends the child's work or talk by asking thought-provoking, open-ended questions--lots of serious conversations with children about their work. ② |  | Skill development is an integral part of thematic projects, not a separate unit of instruction (more so in language than in math.) ①-③ | Wants to make real connections between projects and skills, not just superficial ones, finds that hard. ① |
| More teacher-made activity centers and projects, fewer commercial "kits". ① | (See thematic Curriculum) | "giving kids reason" seems to mean giving them a logical reason to believe what she tells them, i.e., not just her authority or the book's. ① | *Responds to child as a whole person--thoughts, feelings, knowledge, skill--all interrelated. ① |  | Taking big risks in curriculum design-more trust in her ability to design worthwhile learning experiences. ①-③ | Frustrated with "giving kids reason" in math--feels they may not be ready for that, and learning conventions is okay for now. ③ |
| *"Choice time" is integrated with more formal curriculum (no clear dichotomy between work and play) conscious effort to treat self-directed work as serious academics ③ |  |  |  |  |  |  |
| Attempts to develop rules for use of materials that "make sense" to the kids, given the standard of sharing. ① |  |  |  |  |  |  |

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Vicky: Grade 5, School: Private

| Classroom Organization, Time/ Space/Social Structure | Language Curriculum and Instruction | Mathematics Curriculum and Instruction | Interaction with Individuals | Interaction with Groups | Thematic/ Interest Related Curriculum | Self-Reflection |
|---|---|---|---|---|---|---|
| *Many materials are available in the classroom which pose problems to be solved in a variety of ways. ② | *Displays represent the results of children's own experiments, articulated in their own structure of words or numbers. ③ | Poses problems which require children to invent their own methods of solution. ①-② | Intently avoids any judgement that what a child does is "right" or "wrong", but suggests, on a regular basis that they seek other ways of working at a problem. ① | Raises questions in group meetings which have multiple answers. ① | *Builds reading, writing, math practice on activities which involve children in their substance. ①-③ | Much extended awareness of the complexity involved in learning basic concepts. ① |
| *Schedule of the day is such as to allow extended time on interesting activities. ① | *Children write and read their own stories. ①-③ | After child demonstrates that he/she understand an operational process, then she introduces the conventional algorithm. ③ | Asks questions of children that both inquire into their understanding and push them to new levels. ② | Often uses the phrase "some people say..." to get kids to think about something in a new way. ② | *Encourages children to observe phenomenon and articulate their understanding of what they see. ① | More aware of the issues related to not teaching conventional knowledge structures, but not resolved on this question, especially in concerns about children's success in school outside her classroom. ③ |
| *Teacher interacts with children who are actively engaged in chosen projects, asking them questions which get them to look at what they are doing in a new way ② | Books are used that are related to class projects for individual reading. ①-③ | *When child gets an answer, she asks him/her to explain what it means and how they got it. ② | (See Math Curriculum and Instruction) | Focuses the groups' attention on things that are happening in the environment and raises questions for them to think about. ① | Children are taught to use a variety of media for "understanding" the same data (e.g., graphs, poetry, drawing, writing.) ② | Questioning the relationship between understanding concepts and being able to perform well on conventional school tasks like arithmetic. ③ |
| Children are encouraged to talk to one another about what they are thinking and why in all circumstances, and they do it spontaneously. ① |  | *When child seems confused on number problems, she asks them to work out problems using materials, and then connects back to numbers. ③ |  |  |  | Wonders whether her always being non-committal about whether a child is on the right track is a good thing. ③ |
| Chose to teach a multi-age group so that children could see other children's different knowledge and abilities and the various steps involved in learning something. ② |  | *Uses patterns as basis of instruction. |  |  |  |  |

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Suzanne 6/15/81

Observations

Suzanne, personally and classroom-wise, seems least distracted by the Cambridge turmoil.* She didn't mention it at all except when asked a direct question, and she recommended we not talk about it in the teachers' room (where all the other inhabitants were talking about it). Perhaps she has a good reason to think she will not lose her job. But the general upheaval going on around her doesn't seem to penetrate her classroom in the way it does with the others, even Diane who was not RIFFed. A mother on the personnel committee came into her classroom in the middle of the morning to set up an interview with Suzanne (routine, to be done with all magnet teachers). The mother seemed distraught and said she was embarassed; Suzanne handled the intrusion in a friendly and business-like manner.

I spent the morning in Suzanne's class. Our interview was scheduled to occur during recess and it was agreed that if we needed more time, the assistant would take charge of the class, that Suzanne didn't need to be there during "project time." When I arrived, however, Suzanne was ensconced in a Chapter 766 "Core" meeting which lasted almost to recess time: when she returned, she apologized, said the meeting had been called at the last minute, and she couldn't do anything about it. I observed in her room while she was out and the "assistant-teacher" was in charge. The interview, at Suzanne's suggestion, occurred in the classroom during recess and project time. She and I sat on a couch and talked while the children, organized by the "a.t.," were at work around us. Because of the arrangement of Suzanne's room, our conversation was able to be relatively private. I was not, however, able to tape record it. I did take extensive notes, with which both Suzanne and I felt comfortable.

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* At the time of this classroom observation, all non-tenured and 40% of tenured school teachers had been "given notice." A large number of the tenured teachers were eventually rehired but the disturbance was great.

My first impression of Suzanne's classroom was that there were many more "teacher-made" and "child-made" activity centers and displays than there had been two years ago. In my earlier visits to her room, I remember noting that there were many commercial "kits," oriented toward practicing basic skills in reading and math.

One activity area, for example, labelled "cooking" had "directions for chefs, waiters, and waitresses" written out in magic marker on large sheets of paper. They were organizational directions which left the choice of a project up to the participants. A nearby area/display was entitled "making sentences make sense." It showed examples of confusing sentences with the confusions edited out in yellow marker. There were also directions like: "leave out extra words such as 'and then,' 'so,' 'because,' 'and so'" which were illustrated with corrected examples.

This was also written out in the "teacher's hand."

All along one wall was an extensive "work area with directions" for doing book-binding, and on the other side of the room, a similar set up labelled "writing and printing center." These were related to the major project in which the class was currently engaged: writing, editing, and binding their own individual autobiographies. A second thematic area was a "sea life table" with posters of sea animals, shells, and dried seaweed displayed, as well as teacher-made booklets about sea life and several trade books.

When I arrived, two girls and one boy were working on their books (writing and assembling), two boys were playing a commercial board game in an area labelled "math center", two girls were putting together a puzzle, the pieces of which were the 50 states in the US, all of the others were working on packets of papers at their desks or at a round table, with the help and encouragement of the "a.t." who moved from one place to another. These papers included some conventional math and grammar practice, but were primarily oriented toward the autobiographies each child had written. (They were to make lists of nouns, verbs, and adjectives they had used in one or another part of what they had written.)

On the wall, there was a list of the things to be included in the autobiography, as follows:

name poem

name history

recopied and edited draft of autobiography with titles

dedication

title for book

time line

family data page

ideas for illustration

The directions said that these all needed to be collected before beginning the binding process.

Students' desks were arranged around the room among the activity centers, generally in pairs. In front of pairs of desks was a corrugated cardboard wall, which served in many cases as a personal notice board. While I was there, students moved freely around, working at their desks, at activity areas, on the floor and at two free-standing tables. They carried on generally quiet conversations, related to their work or not, and all seemed engaged with one or another activity. Their work seemed directed by a list of jobs on the board as follows:

Musts:

autobiography nouns, verbs, and adjectives

language usage packets (may take break between pages)

any unfinished math assignments

check folders and desks

Choice:

- blocks
- reading
- map study
- life in the sea
- puzzles
- backgammon, chess, etc.
- water color painting
- plant journal,
    - poems
    - growth
- math materials
    - chip trading
    - kalah
    - fraction disks
    - fraction tiles
    - etc.
- sewing

About 2/3 of the children in the room seemed to function quite independently during the hour and a half that Suzanne was out. The others received help with their work, interpretation of directions, and reminders about what they were supposed to be doing from the aide. I observed people working on their books, doing map puzzles, and playing a board game. In all of these cases, there was conversation among the children about the substance and the process of the activity as well as talk about unrelated matters while the activity continued. There were arguments about rules and "what you're supposed to do" which seemed quite typical of children in the fourth grade.

When Suzanne came in, nothing much changed. She did not announce her presence or go around "checking up" on everyone. Yet it was clear that her sense of what everyone was doing was substantial. There was an obvious difference between the help and focus on task that the "a.t." had been giving, and Suzanne's more direct interaction to meet students' differing needs. She seemed to know where they were and where she wanted them to go, yet she was not directive. She sat down with one girl, for example, who was working on her autobiography words. She began at a point in the book where they had obviously "left off" at some prior work period and worked with the girl to edit her writing. She asked the girl to try to find mistakes herself, and also pointed some out. When a word was spelled incorrectly, Suzanne went over some rules until they arrived together at the correct spelling (with Suzanne being the final judge). The word was then written correctly in the child's personal word list book, and corrected in the autobiography (which was written in pencil). There was talk about which words in a title need to be capitalized, with Suzanne asking for the rule and then having the child apply it, and also a talk about the differences between "there," "they're," and "their." In no case did Suzanne say, "This is wrong and here is what it should be." The correction was always arrived at through directed questioning and suggestion.

Suzanne's next interaction was with a boy who came up to her with a card that had a picture of a jellyfish on it. He said it was the same as the picture on a large poster in the "Sea Life" center. Suzanne walked him over to the poster and put the card up next to it and said, "They are sort of the same, but sort of different. Can you see how they are different?" The boy said they looked the same to him. She pointed out that they had different names and suggested he try to look carefully at how they might be different. Then another boy came up and handed her a shoe box with a lot of foreign coins taped inside it. She said, "Oh, is this your collection? What are these?" He said, "Money from another country, my father bought it." She said, "Did he buy it in a coin store?" He said, "No, he was there." She asked, "Where is it from?" He said he didn't know, and she looked over the bills, trying to figure out where they might be from. He did not seem very interested in looking for clues; the ones she proposed made sense to her, but did not engage him. After asking me where I thought they might be from, she told him where the "Collections" were to be displayed and gave him some directions about the process.

The recess bell rang, but many of the kids stayed in the room and had snacks while working on things. Others went in and out through a door which went from the classroom directly outside. Suzanne directed me to the couch where we were to sit and talk. The personnel committee mother came in at this point, and then some girls in the class pulled up their chairs to join this "sewing circle" of grownups. Suzanne asked them to go and do something because we needed to talk, which they did.

## Interview

I asked her at first, just to talk about what was currently going on in her classroom. She said that at the beginning of the year, the curriculum begins with skills and develops into projects, but now, the projects came first and skill development grew out of them. She liked the latter way better and would like to be able to do more of that next year, i.e., to begin right off with projects. I asked where the idea to do the autobiographies came from. She said she had two goals in mind: "to do something that was really 'me' oriented, and to have a big end of the year culmination -- something that was fun _and_ personal." In the spring, she told me, they had done a big research project that was related to Green Acres/ Colonial history. She thought that was also somewhat personal, but not as much as autobiographies. (There were some remnants of that project around the room -- a

3' x 8' fairly sophisticated but child-made scale model of a village, a list of "rules in the 1700's," a list of "punishments in the 1700's," a list of "qualities of a good governor" and an announcement of an election. The first two lists seemed "researched," the last one, constructed by the students.) Suzanne said that the specific content of the autobiography project, including binding them into personal books came from a chapter that Follet Publishers made available for field testing.

The thing that was most interesting to Suzanne in presenting this project was that she wanted to write out all the directions in such a way as to enable the children to work independently in all the centers. She worked very hard at getting down the wording of the direction, trying them out on people to see if she could get across exactly what she meant. This was a large undertaking, she said, because the center involved 5 different "stations" and some of the pieces of the project had as many as 20 steps. She was most intrigued by the fact that words did not seem sufficient to explain some parts of the process, even to adults. She said, "I just had to show them that step, and always have someone there to show the kids. I always thought you would be able to find words if you really worked at it, but that's not true."

This interest in "giving directions," making yourself understood by others, and understanding them, seemed related to Suzanne's view, to what we were talking about in the project. She would like to think about it and talk about it some more.

I told her of my observation that the last time I had visited (over a year ago), there seemed to be more commercial, skills-oriented kits being used, whereas now there were more projects (using her word). She said, "It's always more that way at the beginning of the year. Like when you have a kid read a book first, and then have him do a project on it. You can't just start with a project." She went on to talk about how hard it seems to do "interdisciplinary teaching" and also "really get at skills" -- "you have to know they're getting it." (She seemed to have some new insights into this process, to be saying that she had thought interdisciplinary teaching was one thing, but now sees that it's really something else, and hard to do.) I asked why it was hard to start the year with projects, and she said that you "really need to be creative to get the skills integrated with everything else, especially math. If you're gonna do measurement on something, it's easy, but not if you're gonna make real connections to skills." She said that you could find ways to "tack on" skills, but what was really hard was getting them to be substantially there at the heart of the process.

Since it has always been a concern of hers, I then asked Suzanne to talk about how she was currently teaching mathematics. She said she had a newly developed "conscious awareness" of what a group goes through in learning a particular skill. Her work with a small group on any particular topic "begins with general brainstorming to get at what kids already know, and then moves into working strictly with manipulatives before we do any paper and pencil stuff." She emphasized that it was a "real conversation" and not just "questions and answers" and she said, to me: "We know how hard that is, to have a real conversation" (meaning she and I). This gave me an entree to ask about connections to the project, so I asked her directly how it had influenced the way she taught math. She said, "It really has, but I don't know exactly how to say how." She stressed the importance of "listening to kids and trying to figure out what they don't understand," and then switched to talking about the importance of "the questioning process -- figuring out what they do understand and then connecting up with it." (I wondered if the first formulation was more comfortable to her as a description of what she does while the second formulation was thought to be a more adequate statement of "the party line.")

She went on to say that having "conversations" with kids is the important overall idea from the project that permeates her teaching. She said, "No matter what you are trying to teach, you just have to talk about it with the kids and make them understand it. You have to pay attention to what kids understand and fill in the gaps from what they do understand." I asked how this applied to teaching something specific, like long division. She replied that she went through the same process (of conversation and manipulatives, first) and that it seemed easy for kids to really understand long division; "what's hard to get at is the paper and pencil stuff, how you know which numbers go where."

I then asked if she saw effects of the project in her teaching in areas other than math. She stared off into the room, and was quiet for several moments, and then said, "I think it's an attitude -- a general attitude -- all of us were affected personally in one way or another. I think it's primarily the language usage in the classroom, the kind of talking that goes on around things." She then went on to say, "I always thought of the project in terms of group support, although I'm not sure anyone else looked at it that way, because I need to talk about the philosophy of my classroom. Not talk about it directly, but through the stuff we were doing. Though it seems disconnected, it's an invisible kind of support. Like the moon, it supports the value of talking about things, that this kind of thinking is good, and so you listen to kids. You try to figure out what to say next, you get them to think. We learned to ask questions of other persons, and to express ourselves more clearly. Like you ask a question of a kid to get a question back again, and you find out what kids know by the questions they ask."

I then asked Suzanne what she would think of doing the seminar over again with another group of teachers: which parts seemed most valuable, which could be dropped. She said "It was a real important, invaluable experience for me, but I think you've got to get the right group together. I don't know about stipends and staff development credit, what people's true motivations are." I asked if we should drop those things, and she said, quite firmly, "Oh, no." She said, in her view, "The thinking and learning we did together connected directly to the classroom. For me that was obvious, but others had trouble. I don't know where everyone stands on it. It depends on how willing people are to evaluate themselves. If their intent is a stipend or credit, then it probably couldn't be so good."

Suzanne was beginning to get drawn away from the interview by events in the class, so I asked her only one further, general question. I asked if she could have her ideal teaching job next year, what would it be? She said, straight out, "a third grade alternative classroom in this school but not necessarily in this program." She said she wanted to change programs because she felt as though the one she was in had "a floating structure, didn't really know where it was going." She felt something was missing, but couldn't quite say what. She said she thought she could get more experience in a third grade with "teaching primary skills in interesting ways that were not skill isolated."

I asked her how she felt about all the activity that resulted from proposition 2½. She said that it made her ask herself "do I really want to teach? Or would I rather be doing something else?" And, she decided, "I really love it. And even if I could, I wouldn't be happy just working 8 to 2. This has been a real year of transition for me; the kids may not have got all the skills they were supposed to but they had a good experience and they have good attitudes. I decided to work in the reverse way: instead of limiting the hours I work, I'm going to try to fit the rest of my life in around it. So far, it's not working too well, but I have managed to detach teaching from my weekends."

## VII SKETCHES OF THE PARTICIPANTS' CURRENT ACTIVITIES

### Groups

The year after the project ended, nine of the teachers met together regularly, with rotating presence on the part of the staff, to continue some of the discussions and activities. These meetings went on in alternate weeks throughout the year, and gradually came to focus on two main topics. One was continuing to work together with children, trying to understand their understanding. The other was continuing to work at understanding the motions of the sun and moon and developing models which help explain those motions.

The group that continued to focus on the moon continues to meet now, for the fifth year, every second or third week. It consists of four teachers from the first year's group and two from the second year. In addition to expanding and deepening their understanding of the movements of heavenly bodies, they often move into discussions about teaching, children, and teacher education. The group also serves as a support group, in these times of cutbacks, internal battles within the school system, and shifting jobs.

The group that continued to focus on children has changed emphasis. In the late winter of that third year, one of the teachers came to a meeting after having seen "The Day after Trinity"--the story of J. Robert Oppenheimer and the building of the first nuclear bomb. She was in some consternation. We had been focussing, in our work with children, on math and science activities, on developing independence of thought and interest. In short, she exclaimed, the kind of independence of thought and activity that we were working to develop was brilliantly exemplified by Robert Oppenheimer: curiosity about how things worked, confidence in his ability to figure things out, imagination in conceptualizing alternative solutions, and so on and so on. And what did he do with those abilities? He built the bomb! She urged everyone in the group to see that film, which we did, and to think about what other fundamental tings we wanted to develop in children.

The line was not absolutely direct, from that film to the next stage. Nonetheless, before the end of that school year, the work we had been doing with children stopped, and five of the group had become active in the peace education movement, which was then just beginning in the Cambridge area. Three of these teachers are the core of the curriculum committee of the Cambridge Peace Education Project. They have gained the support of the administration of the Cambridge Public Library system, and the Cambridge Teachers' Association, and have organized two in-service courses for Cambridge teachers, as well as working on the issues in their own classrooms.

Two others have beem members of the curriculum committee of Educators for Social Responsibility. They were two major authors of the K-3 section of the Curriculum and Resource Guide published by that organization, for the first national Day of Dialogue on the nuclear threat, held on October 25.

Of these five, several have organized and led workshops at conferences on this issue. On other occasions, Eleanor Duckworth, who has also been working with these committees, has made presentations based directly on their work with the children in their classes.

## Individuals

First year participants:

Carol: At the end of the first year of the project, Carol had a baby and has not resumed teaching.

Diane: In the middle of the project's first year, Diane had said (rather apologetically) that while she respected those in the group who were informal-classroom teachers, she "Could never feel comfortable" in such a setting. She had argued then, that with her background, schooling, and "kind of person I am," she would always be happier in her "more structured," traditional classroom environment. At the end of the second year of the project, she wrote: "I may as well write this down right at the beginning so that it will be forever recorded: I can no longer teach as I have, things must change!" Diane did make changes in her classroom over the next two years. Then, last spring, after 13 years of teaching, she applied for a position in one of the alternative programs in the Cambridge Public Schools--a move she found very risky, but she made it in order to have support for teaching as she now wanted to.

With thirteen years of tenure, the job she held was secure, but she chose to make this change, anyhow. Diane was impressive enough in her interviews to be offered her choice of positions in three programs and is now teaching in one of them.

Diane is a member of the Moon Group and the Cambridge Peace Education Project.

Helen: At the end of the second year of our project, Helen's small school was closed, and she was transferred to another. At the end of the following year, she was one of the scores of tenured Cambridge teachers to be dismissed as a result of the taxpayers' revolt referendum. She applied, as did dozens of others, for positions in alternative programs, where years of seniority were not the ultimate factor in retaining teachers. Her recommendations and interviews were impressive, and she was asked to join the staff of one of the alternative schools, where she now teaches. She was given notice of dismissal again the following year, but this time all such notices were reversed.

She is a member of the Moon Group and the Cambridge Peace Education Project.

Isabel: Continues to teach in the Cambridge Public Schools.

Jessica: With the continuing cutbacks in the Cambridge Schools, and her relatively few years of tenure, Jessica has been given notice of dismissal each of the last two years. However, the first year teachers in alternative programs were retained for reasons of qualifications as well as seniority, and the second year all notices were evenutally reversed. She continues to teach.

Jessica is a member of the Moon Group and the curriculum committee of Educators for Social Responsibility.

Lee: Since Lee did not have tenure, she was not rehired in the Cambridge Schools. She is continuing her work as a graduate student in education.

Suzanne: Like Jessica, Suzanne continues to teach in an alternative porgram, after receiving notice of dismissal two years in a row.

She is a member fo the Moon Group and the Cambridge Peace Education Project.

Second year participants:

Anna: Continues to teach in the Cambridge Public Schools.

Deborah: Has been on maternity leave since the end of the project, having had two babies in that time. She wants to return to teaching in the fall of 1983.

Heidi: Was not yet tenured in Cambridge, and has therefore not been able to teach there since the end of the project. She badly wanted to teach again, and finally secured an administrative position in a private school, which gave her the opportunity to do some substitute teaching. This year, she is teaching half-time in that same school, as a resource teacher.

Heidi came, to begin with, to the group meetings after the project ended. She found it too painful to continue, however, talking about teaching when she so badly wanted to teach, but couldn't.

Karen: Took a leave from teaching to spend a year doing psychological research. The year she wanted to return was the year of the cutbacks, and she received notice of dismissal. She was able to continue her "on-leave" status for two further years, and is hoping she will be able to return to teaching in the fall of 1983.

She is a member of the Moon Group, and finds it important, among other things, as a way of keeping in touch with teaching.

Katherine: Continues to teach in the Cambridge public schools.

Ruth: Has left teaching to enter the field of family therapy, which she sees as a continuation of the work she did as a teacher and in this project.

Sara: Continues to teach in the Cambridge Public Schools.

Vicky: Was given notice of dismissal. Believing that the chances of reversal of that notice were very low, she accepted the offer of a job to teach in a private school in Cambridge, where she now is.

She is a member of the Moon Group and the curriculum committee of Educators for Social Responsibility.

Summarizing the current picture of our 15 participants (7 in the first-year group, 8 in the second) we see the following: nine participants are still teaching full-time, two are hoping to be re-hired full-time in the fall of 1983, and one is teaching part-time, while seeking a full-time position. Of the six who are not currently teaching full-time, two left to have babies, three were not re-hired as a result of cutbacks, and one left to develop her career in a different direction.

It is clear that the project had a significant impact on the six teachers who continue to meet together in one or both of the two groups described above. (The five of those who are still teaching are described in Section VI.) In addition, we are in occasional contact with five others, who always make warm and specific references to the experience. While the project clearly did not have as profound an effect on them, they do report that it had an influence on their subsequent teaching. (See, for example, the quotes from the final interviews, at the end of Section IV.)

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